STABILITY UNDER POWERS OF MINSET OF HYPERBOLIC IRREDUCIBLE AUTOMORPHISM

by

Sonya Leibman

A dissertation submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematcis

The University of Utah

August 2014

The University of Utah Graduate School

STATEMENT OF DISSERTATION APPROVAL

The dissertation of **Sonya Leibman**

has been approved by the following supervisory committee members:

Mladen Bestvina	, Chair	**4/22/14** Date Approved
Kenneth Bromberg	, Member	**5/16/14** Date Approved
Jonathan Chaika	, Member	**4/22/14** Date Approved
Patrick Reynolds	, Member	**4/22/14** Date Approved
Yair Glasner	, Member	**4/22/14** Date Approved

and by **Peter Trapa** , Chair/Dean of

the Department/College/School of **Mathematics**

and by David B. Kieda, Dean of The Graduate School.

ABSTRACT

We show that in Outer Space, the minset of the displacement function of a hyperbolic irreducible automorphism eventually stabilizes under further powers if and only if no train track representative of the automorphism has a Pre-Nielsen Path. We then analyze what automorphisms of different ranks and indices have stable minsets, showing that almost every index of automorphism has examples with an eventually stable and never stable minset.

CONTENTS

LIST OF FIGURES

NOTATION AND SYMBOLS

Throughout this paper, we will adopt the following shorthand notation.
A: inverse of a generator a in F_n
$\dot{a}$: part of an edge with the marking a
D_φ: map from $T_v(\Gamma)$ to $T_{\varphi(v)}(\Gamma')$ induced by $\varphi : \Gamma \longrightarrow \Gamma'$
$\bar{e}$: an oriented edge or a path e traversed backwards
$\tilde{f}$: displacement function of f from $\mathcal{X}_n$ to $[0, \infty)$ given by $\tilde{f}(\Gamma) = d(\Gamma, \Gamma \cdot f)$.
$f_\Gamma(\mu)$: image of a loop μ under an optimal map $\varphi : \Gamma \longrightarrow \Gamma$ induced by f
F_n: free group on n generators
$f_S(\mu)$: image of loop μ under an optimal map induced by f for a graph Γ in simplex S.
$i(f)$: index of f
$IW(f)$: ideal Whitehead graph of f
Λ: attracting lamination
$\mathcal{L}_\varphi$: axis in Outer Space corresponding to train track representative $\varphi : \Gamma \longrightarrow \Gamma$.
$\mathcal{L}_f$: an axis in Outer Space corresponding to hyperbolic irreducible automorphism f
$l_\Gamma(\rho)$: length of path ρ in graph Γ
$LW(v)$: local Whitehead graph at v
$M(f)$: minset of the displacement function $\tilde{f} : \mathcal{X}_n \longrightarrow [0, \infty)$ given by $\tilde{f}(\Gamma) = d(\Gamma, \Gamma \cdot f)$.
NP: Nielsen Path
NPE: Nielsen Path equivalence class
PNP: Pre-Nielsen Path
R_n: rose, a bouqet on n circles
$\sigma_{\Gamma,f}(\mu)$: stretch of the loop μ in Γ under f_Γ, i.e. $\dfrac{l_\Gamma(f_\Gamma(\mu))}{l_\Gamma(\mu)}$
$SW(v)$: local stable Whitehead graph at v
$T_v(\Gamma)$: set of directions at point v in Γ
$\mathcal{X}_n$: Outer Space

ACKNOWLEDGMENTS

Thank you to my advisor Mladen Bestvina for all his help, expertise, and patience. Thank you to everyone else who gave support, advice, and mathematical help.

CHAPTER 1

INTRODUCTION AND BACKGROUND

1.1 Introduction

Motivated by the Teichmüller approach to the study of Mapping Class Groups, in 1986, Culler and Vogtmann defined Outer Space in [CV86] for the study of $Out(F_n)$. For the definition and details, see Section 1.2.1. Outer Space is a $(3n-4)$-dimensional simplicial complex on which $Out(F_n)$ acts with finite point stabilizers. Bestvina and Handel showed the existence of train track maps representing outer automorphisms. Then the notion of distance was introduced and it was shown that the action of $Out(F_n)$ on $\mathcal{X}_n$ splits automorphisms into three categories: elliptic, hyperbolic, and parabolic. See [Bes11] for a laconic proof of this following the lines of Bers's proof of the classification of mapping classes. In this paper, we will be looking at hyperbolic automorphisms, namely those whose displacement function achieves its infimum, and the infimum is not zero. The minset $M(f)$ of the displacement function of a hyperbolic irreducible automorphism f (see Section 1.2 for the definitions) is an object of study and conjecture. However, it turns out that the minset might change as we take powers of an automorphism. In many cases, when we study an automorphism, we need to take powers, whether just a few to make sure it is rotationless, or many to study laminations, Perron-Frobenius eigenvalues, and limiting behaviors. Therefore, it is important to see what happens to the minset as we take powers and whether it eventually stabilizes under further powers. This is the question we attempt to answer. In fact, we show the existence of examples where the minset never stabilizes.

Theorem 1.1 *Let f be a hyperbolic irreducible automorphism. There exists a k such that $M(f^{mk}) = M(f^k)$ for all $m \in \mathbb{N}$ if and only if no train track representative of f has a Pre-Nielsen Path.*

With powers, the Nielsen Path to which the Pre-Nielsen Path maps has more and more cancellation, so we can fold the illegal turn of the Pre-Nielsen Path further and further, while staying in the minset. Hence the minset increases with powers and never stabilizes. See Figure 1.1 for an illustration of Pre-Nielsen and Nielsen Paths.

Note that the possibility of a never stable minset for a hyperbolic automorphism in Outer Space is in contrast with the Teichmüller Space case. Bers showed that the minset of the displacement function of a pseudo-Anosov automorphism in Teichmüller Space is always a line, so it does not change with powers of the automorphism.

However, there are other examples of groups acting on spaces with hyperbolic elements with a never stable minset. Thank you to Yair Glasner for providing the following example. Consider the arithmetic group $G = SL_2(\mathbb{Z}[\frac{1}{3}])$. We can think of it as a lattice in the group $SL_2(\mathbb{Q}_3) \times SL_2(\mathbb{R})$. Let X be the space $T \times \mathbb{H}^2$ where T is the 4-valent tree and $\mathbb{H}^2$ is the hyperbolic plane. Then $SL_2(\mathbb{Q}_3) \times SL_2(\mathbb{R})$ has a natural action on X which induces an action of G on X. Now consider an element A of G such that A has integer entries as a matrix and $tr(A) > 2$, for instance

$$\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

Since $tr(A) > 2$, A is hyperbolic on $\mathbb{H}^2$, and since A has integer entries, it is elliptic on T. Now we consider what this means for A on the whole space X. Since A is elliptic on T, it fixes at least one point t_0 in t. Let $Fix_T(A^n)$ be the subset of T fixed by A^n. Let B_k be a ball of radius k centered at t_0. There are only finitely many points in B_k, so there is a power A^{n_k} of A such that A^{n_k} fixes B_k (by this we mean pointwise fixes). This means that for any $m \in \mathbb{N}$, A^{mn_k} fixes B_k. Let $m \in \mathbb{N}$. Since A^m is not the identity, there is a k s.t. A^m does not fix B_k. However, A^{mn_k} fixes B_m, so $Fix_T(A^{mn_k}) \neq Fix_T(A^m)$, meaning there is no power of A such that its fixed point set in T stabilizes under further powers. Since A is hyperbolic on $\mathbb{H}^2$, it has an axis S of points with minimal displacement under A^n. Let $M(A^n)$ be the subset of X minimally displaced by A_n. Then $M(A^n) = Fix_T(A^n) \times S$. Since there is no power of A s.t. $Fix_t(A^n)$ is stable under further powers, there is no power of A s.t. $M(A^n)$ is stable under further powers.

We need some preliminary results that will be used in the main proofs. These are found in Chapter 2. First, using primarily results from [AK08], we show that there is a finite neighborhood of an axis of a hyperbolic irreducible automorphism g that contains the minset of any power of g. This allows us to focus on a finite number of g-orbits of simplices. Using this, we come up with a power of g that is rotationless and sends candidate loops to loops that are legal except for the illegal turns taken by Nielsen Paths they contain (see Section 1.2 for all the definitions). This is the power that will potentially have a minset that is stable under further powers.

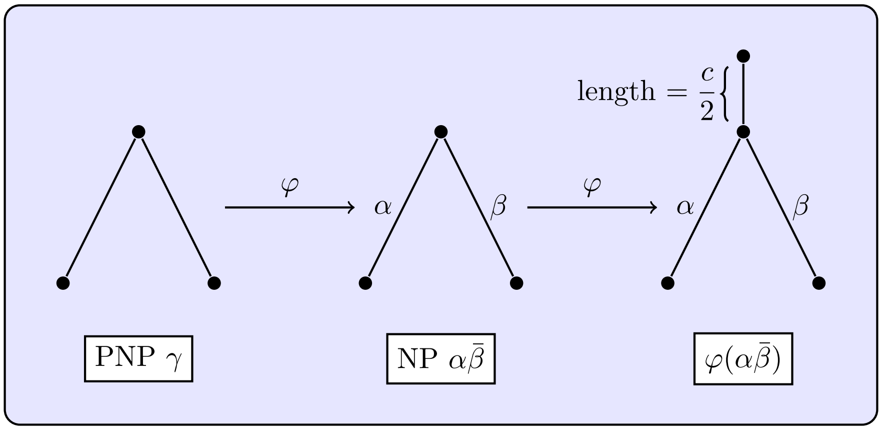

Figure 1.1. Pre-Nielsen Path and Nielsen Path

In Chapter 3, we prove that the minset of a hyperbolic irreducible automorphism f eventually stabilizes if and only if no train track representative of f contains a Pre-Nielsen Path. This involves three different theorems. In Theorem 3.1, we show that if f has a representative with a PNP, its minset never stabilizes. This is achieved by folding the illegal turn of the PNP to get new representatives in the minset. As we take powers of f, the NP to which the PNP maps has more and more cancellation, so we can fold longer and longer segments of the PNP. However, the total amount we can fold has an upper bound which is never achieved, so the minset never stabilizes. The next result is in Theorem 3.2, where we show that if no representative of f has an NP, then the power we found in Chapter 2 has a stable minset. In Theorem 3.3, we show that if representatives of f might have NPs but no PNPs, there is a power with a stable minset. We do this by showing that for any simplex in the minset, for every possible combination of NPs, there is power such that any other graph in the same simplex with the same NP combination is in the minset of the same power. Since there are finitely many types of simplices and NP combinations, we get a power with a stable minset.

Next, we need to figure out which hyperbolic irreducible automorphisms have representatives with or without NPs and PNPs. Currently, the main way of distinguishing hyperbolic irreducible automorphisms is by splitting them into three categories: geometric, parageometric, and ageometric, and then by further differentiating ageometric automorphism using indices $1.5 - n \leq i \leq 0$. In Chapter 4, we analyze which of these fall into which NP categories. Somewhat anticlimactically, it turns out that almost every rank and index has examples of all three types of automorphisms: no NPs, NPs but no PNPs, and NPs with PNPs. This is shown using specific examples listed in Chapter 6. The chapter contains an extensive list of examples necessary for the proofs, but hopefully they will be useful as a reference for other results, since there are very few concrete examples in current literature. We also show that for any hyperbolic irreducible automorphism, if we only consider stable train track representatives, the minset does eventually stabilize.

Corollary 1.1 *Let g be a hyperbolic irreducible automorphism. Let $SM(g)$ be the set of stable representatives in $M(g)$. Then there is an automorphism f, a power of g, s.t. $\forall k$, $SM(f^k) = SM(f)$.*

An extension that would make these results much more applicable would be some method of showing that a hyperbolic irreducible automorphism has no representative with PNPs. It is simple to show this locally, but since PNPs are not composed of stable directions, they do not show up in Stable Whitehead or Ideal Whitehead graphs, which are the main way of analyzing hyperbolic automorphisms.

In 5, we show some results that ended up not being part of the primary subject of the paper. The main one shows that the minset of a hyperbolic irreducible auotomorphism is contractible in a simplex of $\mathcal{X}_n$. We also address the connection between this paper and several questions posed in [HM08].

1.2 Background

Here is a brief survey of Outer Space and other definitions used in this paper. Throughout, we will give references to more detailed descriptions of the concepts.

1.2.1 Outer Space

For a detailed survey of Outer Space, see [Vog] and [Bes12].

Let $F_n =< x_1, x_2, ..., x_n >$ be the free group on n generators. Let the **rose** R_n be a bouquet of n circles with oriented edges labeled by $e_1, ..., e_n$ and vertex v. Now identify $\pi_1(R, v)$ with F_n by letting x_i be the homotopy class of the loop e_i. A **graph** is a cell complex of dimension less than or equal to 1. A **marked graph** (Γ, τ) is a graph Γ equipped with a homotopy equivalence $\tau : R_n \longrightarrow \Gamma$ called the **marking** of (Γ, τ). The homotopy equivalence τ induces an isomorphism $\tau_* : F_n \longrightarrow \pi_1(\Gamma, \tau(v))$. Two marked graphs $\tau : R_n \longrightarrow \Gamma$ and $\tau' : R_n \longrightarrow \Gamma'$ are **equivalent** if there is a homeomorphism $\varphi : \Gamma \longrightarrow \Gamma'$ such that $\varphi\tau$ is homotopic to τ'. In practice, it is more convenient to use the **inverse of a marking**, i.e., a homotopy equivalence $\Gamma \longrightarrow R_n$. The inverse marking can be defined by specifying a maximal tree T in Γ, orienting all edges in $\Gamma - T$, and labeling them with a (possibly different) basis of F_n, expressed as words in $x_1, x_2, ..., x_n$. This choice defines a map $\Gamma \longrightarrow R_n$ by collapsing T to a point and sending each edge to the edge path specified by the label. Figure 1.2 shows an example.

A **metric** on a finite graph is an assignment of nonnegative numbers $l(e)$, called lengths, to the edges e of Γ. This allows us to assign **lengths of paths** in Γ. A graph with a metric is called a **metric graph**. The volume of a finite metric graph is the sum of the lengths of its edges.

Consider a finite metric marked graph as a triple (Γ, l, τ), where Γ is a finite graph, l is a metric on Γ with volume 1, and $\tau : R_n \longrightarrow \Gamma$ is a marking. Two triples (Γ, l, τ) and (Γ', l, τ') are **equivalent** if there is an isometry $\varphi : \Gamma \longrightarrow \Gamma'$ such that $\varphi\tau \simeq \tau'$.

Outer Space, call it $\mathcal{X}_n$, is the set of equivalence classes of metric marked graphs of volume 1 and vertex valence of at least 3, i.e., $\mathcal{X}_n = \{(\Gamma, l, \tau)\}/\sim$. We will usually just talk about points $\Gamma \in \mathcal{X}_n$, omitting l and τ from the notation.

We use the metrics on graphs to define simplices in $\mathcal{X}_n$. Suppose Γ is a graph with k edges and $\tau : R_n \longrightarrow \Gamma$ a marking. Then the set of possible metrics on Γ gives an open simplex $S(\Gamma)$ of dimension $k-1$ by $\{(l_1, l_2, ...l_k) | l_i > 0, \sum_{i=1}^{k} l_i = 1\}$. Suppose T is a forest in Γ. Let Γ' be the graph obtained from Γ by collapsing all edges of T to points. Now $S(\Gamma')$ can be identified with the open face of $S(\Gamma)$ which corresponds to the edges in T having coordinate 0. A **simplex** in $\mathcal{X}_n$ is the simplex with missing faces obtained from the union of $S(\Gamma)$ and all the faces corresponding to T ranging over all forests in Γ.

Now, we will define a nonsymmetric **distance** on outer space. If $[(\Gamma, l, \tau)]$ and $[(\Gamma', l', \tau')] \in \mathcal{X}_n$, a map $\varphi : \Gamma \longrightarrow \Gamma'$ is a **difference of markings** if $\varphi\tau \simeq \tau'$. Only consider Lipshitz maps with constant slope on each edge, and let $\sigma(\varphi)$ be the Lipshitz constant of the map φ. Let $d(\Gamma, \Gamma') = \inf_{\varphi} \log(\sigma(\varphi))$. This definition of distance satisfies the triangle inequality and is positive definite, but is not symmetric. By Arzela-Askoli, the infimum is actually achieved. A difference of markings φ is **optimal** if $\sigma(\varphi) = d(\Gamma, \Gamma')$.

Let φ be an optimal map. Let μ be a loop in Γ. Any time we talk about loops, we mean **tightened** loops, meaning that there is no backtracking. Now, $\varphi(\mu)$ is also a homotopy class of loops in Γ, and we reserved that notation for the tightened version of the image loop. Let $\sigma(\mu) = \dfrac{l_{\Gamma'}(\varphi(\mu))}{l_\Gamma(\mu)}$ be the **stretch** on μ. Then $d(\Gamma, \Gamma') = \sup_{\mu} \log(\dfrac{l_{\Gamma'}(\varphi(\mu))}{l_\Gamma(\mu)})$, and the supremum is realized (see [Bes11] for a proof).

For any $\Gamma, \Gamma' \in \mathcal{X}_n$, there is a maximally stretched loop μ that is one of the following:

- an embedded circle
- a wedge of two embedded circles

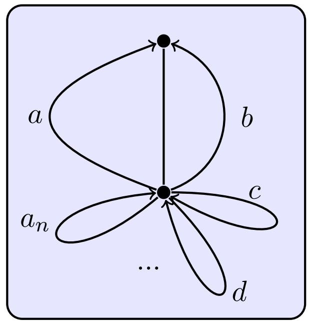

Figure 1.2. Marked graph

- a barbell, i.e., a concatenation $\gamma_1\gamma_2\gamma_3\bar{\gamma}_2$ where γ_1 and γ_3 are disjoint embedded circles, and γ_2 is a path that only intersects γ_1 and γ_3 at one of its endpoints.

Call a loop like this a **candidate** loop. The number of edges of a candidate loop is bounded by $3n-3$, where n is the rank of F_n, so Γ has a finite number of candidate loops. See [AK08] for details. If $\varphi : \Gamma \longrightarrow \Gamma'$ is a difference of markings, there is a candidate loop μ in Γ such that $d(\Gamma, \Gamma') = \log(\frac{l_{\Gamma'}(\varphi(\mu))}{l_\Gamma(\mu)})$.

For $\Gamma \in \mathcal{X}_n$, a **direction** at a point v in Γ is a germ of geodesic paths $[0, \epsilon] \longrightarrow \Gamma$ that send 0 to v (this is the same as the beginning of an oriented edge with initial vertex v). Let $T_v(\Gamma)$ be the set of direction at v. Note that if v is not a vertex, then $T_v(\Gamma)$ contains two directions.

Now an optimal map $\varphi : \Gamma \longrightarrow \Gamma'$ induces a map (one can think of it as a sort of derivative) $D_\varphi : T_v(\Gamma) \longrightarrow T_{\varphi(v)}(\Gamma')$.

1.2.2 Action of $Out(F_n)$ on $\mathcal{X}_n$

For details, see [Bes11].

We can define an action of $Out(F_n)$ on $\mathcal{X}_n$. Think of $f \in Out(F_n)$ as a homotopy equivalence $h : R_n \longrightarrow R_n$. Then the action is $(\Gamma, l, \tau) \cdot f = (\Gamma, l, \tau h)$. The homotopy class of τh is independent of the conjugacy class of f, so the action is well defined.

We can think of f as giving an optimal map $f_\Gamma : \Gamma \longrightarrow \Gamma$ such that for a loop μ, $f_\Gamma(\mu) = \tau h(\tau^{-1}(\mu))$ (note that f_Γ might not be the unique choice of such a map, but f_Γ of a loop is well defined, since we only consider tightened loops). So for a specific choice of optimal map f_Γ, the image of a path ρ is well defined, but depends on the choice. However, for a loop μ, $f_\Gamma(\mu)$ is well defined and does not depend on the choice of f_Γ. Note also that if Γ' is a graph in the same simplex but with a different metric, if we ignore the metric, $f_{\Gamma'}(\mu) = f_\Gamma(\mu)$. So in a simplex S, we will use the notation $f_S(\mu)$ for this image.

The **stretch under** f **in** Γ of a loop μ is $\sigma_{\Gamma,f}(\mu) = \dfrac{l_\Gamma(f_\Gamma(\mu))}{l_\Gamma(\mu)}$.

We call $f \in Out(F_n)$ **reducible** if there are proper free factors $F_1, \ldots, F_k$ of F_n s.t. f transitively permutes the conjugacy classes of the F_is and such that $F_1 * \cdots * F_k$ is a free factor of F_n. If f is not reducible, we call it **irreducible**.

Furthermore, if every power of f is irreducible, we call f **fully irreducible**.

Consider the **displacement function** $\tilde{f} : \mathcal{X}_n \longrightarrow [0, \infty)$ given by $\tilde{f}(\Gamma) = d(\Gamma, \Gamma \cdot f)$.

The automorphism f is **hyperbolic** if $\inf \tilde{f} > 0$ and it is realized.

Throughout this paper, we will only consider hyperbolic fully irreducible automorphisms.

The **stretch constant** λ of f is $e^{\inf \tilde{f}}$. So $\log(\lambda)$ is the minimal displacement by f of any point in $\mathcal{X}_n$.

The **minset** $M(f)$ is the set of points Γ in $\mathcal{X}_n$ such that $\tilde{f}(\Gamma) = \log(\lambda)$. This also means that for $\Gamma \in M(f)$, $\lambda = \max\limits_{\mu,\text{ candidate loop in }\Gamma} \sigma_{\Gamma,f}(\mu)$.

Let $\varphi : \Gamma \longrightarrow \Gamma$ be an optimal map in the homotopy class given by f. Now $D_\varphi : \bigcup\limits_{v \in v(\Gamma)} T_v(\Gamma) \longrightarrow \bigcup\limits_{v \in v(\Gamma)} T_{\varphi(v)}(\Gamma)$ is a map on the directions in Γ. Note that φ might not send vertices to vertices, so the two unions might be different. Since we defined $T_p(\Gamma)$ for any point in Γ, the map still makes sense.

1.2.3 Train Tracks

Good references for the construction of train tracks and the proof of their existence are [Bes11] and [BH92].

Let Γ be a graph. A pair of oriented edges $\{e_1, e_2\}$ in Γ is a **turn** if e_1 and e_2 have the same initial point. We use the notation $\bar{e}$ to represent the edge e with the opposite orientation. We say that a path ρ **crosses the turn** $t = \{e_1, e_2\}$ if $\rho = ...\bar{e}_1 e_2...$ or $\rho = ...\bar{e}_2 e_1...$. A turn is **nondegenerate** if it is defined by distinct edges. Otherwise, it is called **degenerate**.

Let Γ be a graph. A **train track** structure on Γ is an assignment to each turn in Γ a label of **legal** or **illegal**. An path in Γ is a **legal path** if it is immersed and if every turn it crosses is legal. A map $\varphi : \Gamma \longrightarrow \Gamma$ is called a **train track map** if it sends legal turns to legal turns and edges to legal paths. If φ is an optimal map in the homotopy class induced by an automorphism f, then $\varphi : \Gamma \longrightarrow \Gamma$ is called a **train track representative** of f.

As shown in [Bes11], for any hyperbolic irreducible automorphism f with stretch constant λ, there exists a map $\varphi : \Gamma \longrightarrow \Gamma$ in the homotopy class induced by f such that φ is a train track representative. Furthermore, if $\varphi : \Gamma \longrightarrow \Gamma$ is a train track representative, then $\Gamma \in M(f)$. The converse is also true. As shown in Theorem 1.7 in [BH92], if $\Gamma \in M(f)$, and $\varphi : \Gamma \longrightarrow \Gamma$ has stretch constant λ, then φ is a train track map. So every point in the minset has a train track representative.

Throughout this paper, we will mostly be using the minimal train track structure, which is defined in Section 1.2.4.

From now on, unless otherwise stated, every graph $\Gamma \in M(f)$ will be endowed with the minimal train track structure. Thus, it will make sense to talk about legal and illegal turns and loops.

1.2.4 Laminations

For a more detailed explanation of laminations as well proofs of the results used here, see [BFH97].

Let f be a hyperbolic irreducible automorphism with stretch constant λ. Let $\varphi : \Gamma \longrightarrow \Gamma$ be a train track map in the homotopy class induced by f. Take a power of φ if necessary, so that there is a fixed point p in the interior of an edge. Let I be an ϵ neighborhood of p such that $I \subset f(I)$. Now choose an isometry $l : (-\epsilon, \epsilon) \longrightarrow I$. This extends uniquely to an isometric immersion $l : \mathbb{R} \longrightarrow \Gamma$ such that $\forall m \in \mathbb{N}, t \in \mathbb{R}$ we get $l(\lambda^k t) = \varphi^m(l(t))$. The **stable lamination on** Γ, from now on just called lamination Λ, is the equivalence class of isometric immersions containing any immersion obtained by iterating a neighborhood of a periodic point as above. A **leaf** of Λ is any immersion representing Λ. A **leaf segment** of Λ is a segment of some leaf of Λ.

We can define a **minimal train track structure** on Γ by letting a turn be legal if and only if it is crossed by a leaf of Λ. With this train track structure, φ is a train track map.

Note that with the minimal train track structure, a turn t is legal if and only if there exists a loop σ such that $\exists M \in \mathbb{N}$ such that $\forall k \geq M$, $\varphi^k(\sigma)$ crosses t.

1.2.5 Folding

Here we will define the traditional notion of folding (see [Sta91] for details), as well as an extended formal notion of folding that will be used in this paper.

Let $\varphi_0 : \Gamma_0 \longrightarrow \Gamma_0$ be a train track map. Suppose $t = \{e_1, e_2\}$ is an illegal turn with shared vertex v. Consider e_i as a map $[0, l_{\Gamma_0}(e_i)] \longrightarrow \Gamma_0$ with $e_i(0) = v$. Let $s = \max\{x | \forall x' \leq x : \varphi(e_1(x')) = \varphi(e_2(x'))\}$. Then **folding** the turn t for a length $x \leq s$ means identifying $e_1(x)$ with $e_2(x)$, i.e., we are identifying the initial segments of length x of e_1 and e_2. We then rescale the resulting graph by diving the lengths of all edges by $1 - x$, thus getting a graph Γ_x of volume 1. We can define $\varphi_x : \Gamma_x \longrightarrow \Gamma_x$ by sending a point p to $[r \circ \text{fold} \circ \varphi_0(\text{fold}^{-1} \circ r^{-1}(p))]$, where r stands for rescaling. This map is well defined, since rescaling is invertible and for $x' \leq x$, $\varphi_0(e_1(x')) = \varphi_0(e_2(x'))$. Furthermore, one can check that φ_x is a train track map and if φ_0 represents an automorphism f, then so does φ_x. We can use a sequence of folds to construct a **folding path** from Γ to $\varphi(\Gamma)$, which happens to be a geodesic and have all sorts of other wonderful properties and applications. We will not go into detail here, because we will only use a folding path once in order to construct an axis of f.

As mentioned above, we will formally extend the notion of folding here, because we want to keep folding even if we leave the minset. Let $\varphi_0 : \Gamma_0 \longrightarrow \Gamma_0$ be a train track map representing a hyperbolic irreducible automorphism f. Let t be a turn (not necessarily illegal) at a vertex v. Let α and β be rays with initial point v s.t $\bar{\alpha}\beta$ crosses the turn t. Now, to **fold turn t a distance** x, we identify the initial segments of length x of α and β. We then rescale by dividing the length of every edge by $1-x$ to get a graph Γ_x of volume 1. Since $\varphi_0(\alpha)$ and $\varphi_0(\beta)$ might coincide on a length less than x or not at all, we can no longer extend φ_0 to Γ_x. However, f still can be represented by an optimal map $\varphi_x : \Gamma_x \longrightarrow \Gamma_x$, and the image of loops under this map does not depend on the choice of φ_x (since when we tighten the image, we get a unique loop). So f_{Γ_x} of a loop is still well defined. Furthermore, if we consider loops without the metric, then the image of a loop in Γ_0 under φ_0 and the same loop in Γ_x under f_{Γ_x} is the same (if Γ_0 and Γ_x are in different simplices, just split the fold into steps, one for each simplex).

1.2.6 Axis

There is a detailed explanation of the construction of axes and proofs about their properties in [AK08].

Let f be a hyperbolic irreducible automorphism with stretch constant λ. Let $\varphi : \Gamma_0 \longrightarrow \Gamma_0$ be a train track representative. Let $G : [0, \log(\lambda)] \longrightarrow \mathcal{X}_n$ be the folding path from Γ_0 to $\Gamma_0 \cdot f$, parametrized according to arc length. To extend the path, for $t \in \mathbb{R}$, let $k = \left\lfloor \dfrac{t}{\log(\lambda)} \right\rfloor$. Define $G(t) = G(t-k) \cdot f^k$, so we are translating $G([0, \log(\lambda)])$ by f^k and f^{-k}. Let $\mathcal{L}_\varphi = Im(G)$. We call $\mathcal{L}_\varphi$ an **axis of** f.

As explained in [AK08], there is a **nearest point projection** $\pi : \mathcal{X}_n \longrightarrow \mathcal{L}_\varphi$, which has the property that $\pi(f(\Gamma)) = f(\pi(\Gamma))$.

1.2.7 Nielsen Paths

For Nielsen Paths and most of the results about them, see [BH92].

Let $\varphi : \Gamma \longrightarrow \Gamma$ be a train track map. Let p_1 and p_2 be periodic points of φ such that $\varphi^k(p_i) = p_i$. A path ρ between p_1 and p_2 is a **Periodic Nielsen Path** if $\varphi^k(\rho) \simeq$

ρ rel endpoints. A Nielsen Path is **indivisible** if it cannot be written as a nontrivial concatenation $\rho = \rho_1 \cdot \rho_2$, where ρ_1 and ρ_2 are periodic Nielsen Paths. In this paper, we will deal with rotationless automorphisms (defined in Section 1.2.8 and existence proven in [HM08]), which means all Periodic Nielsen Paths will have period one. From now on, an **NP** will be an indivisible Nielsen Path of period one.

A **Pre-Nielsen Path** is a path ρ that is not a Nielsen Path, but such that $\exists k$ with $\varphi^k(\rho)$ a Nielsen Path. From now on, we will call it a **PNP**.

By Lemma 3.4 of [BH92], an NP ρ contains exactly one illegal turn. There are unique legal paths α, β, and τ, such that $\rho = \alpha \cdot \bar{\beta}$, $\varphi(\alpha) = \alpha \cdot \tau$, $\varphi(\beta) = \beta\tau$, and such that $\{\bar{\alpha}, \bar{\beta}\}$ is a nondegenerate illegal turn. Figure 1.3 illustrates this.

Suppose $\varphi : \Gamma \longrightarrow \Gamma$ is a train track representative of a hyperbolic irreducible automorphism f. We can **stabilize** Γ as described in [BH92]. For any NP $\rho = \alpha\bar{\beta}$, one can fold Γ at ρ by folding at the illegal turn of ρ, producing another train track representative $\varphi' : \Gamma \longrightarrow \Gamma$, a quotient map $q : \Gamma \to \Gamma$ satisfying $q\varphi = \varphi q$, and an induced NP $\rho' = q(\rho)$. This is accomplished by defining q to be the fold map which identifies the longest pair of terminal segments of α and β that are identified by φ and are each contained in a single edge of Γ. We say that φ_0 is **stable** if for any sequence of train track maps $\varphi_0, \varphi_1, ..., \varphi_{m-1}, \varphi_m$ such that φ_m is obtained by folding φ_{m-1} at an NP, the fold map $\varphi_{m-1} \longrightarrow \varphi_m$ is not a partial fold (this simply means that at least one of the two segments that is folded consists of an entire edge). As explained in [BH92], after a sequence of folds, we will get a stable representative $\varphi_0 : \Gamma_0 \to \Gamma_0$. Stability of φ_0 implies that φ_0 has at most one indivisible NP up to reversal. If φ_0 does not have an NP, then f is **ageometric**. If φ_0 does have an NP ρ, then if ρ is closed, then f is **geometric**, and if ρ is not closed, then f is **parageometric**.

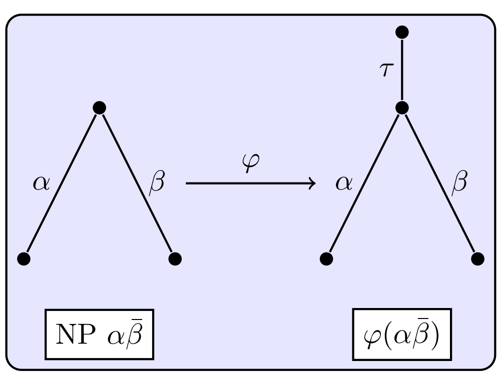

Figure 1.3. Nielsen Path

The properties above are not the definitions of **geometric**, **parageometric**, and **ageometric**. This is just a practical way to distinguish them. For the actual definitions, see [MH08]. Note that ageometric is a less widely used term, but it just means an automorphism that is not geometric or parageometric.

1.2.8 Rotationless Automorphisms

Let $\varphi : \Gamma \longrightarrow \Gamma$ be a train track representative of a hyperbolic irreducible automorphism f. Then v is a **principal vertex** if there are at least three periodic directions at v (i.e., $d \in T_v(\Gamma)$ such that $\exists k \in \mathbb{N}$ with $(D_\varphi)^k(d) = d$) or v is the end of a Periodic Nielsen Path. If every principal vertex is fixed by φ and every periodic direction at a principal vertex is fixed by D_φ, we call φ a **rotationless train track map**. As explained in [HM08] and proven in Proposition 3.24 of [FH09], there is a way to define a **rotationless automorphism** f such that f is rotationless if and only if it has a rotationless train track representative. We will not go into that definition here, since that would involve introducing a lot of new terminology. All we need to know is that for a hyperbolic irreducible automorphism g, there is a power f of g (called a rotationless automorphism) such that every train track map representing f is rotationless. This means every periodic direction is fixed and every NP has period one.

1.2.9 Whitehead Graphs

A great reference for Whitehead graphs and their properties is [HM08]. The reference also contains several results about Whitehead graphs and Nielsen Paths used in this paper.

Consider a hyperbolic irreducible automorphism f. Let $\varphi : \Gamma \longrightarrow \Gamma$ be a train track automorphism. Let v be a point in Γ. Then the **local Whitehead graph** at x denoted by $LW(v)$ is the graph whose vertices are points in $T_v(\Gamma)$, i.e., directions at v, and there is an edge between d_1 and d_2 if there is an edge e in Γ and a $k \in \mathbb{N}$ such that $g^k(e)$ crosses $\bar{d}_1 d_2$. Note that the map D_φ (defined in Section 1.2.1) restricted to $T_v(\Gamma)$ gives a map $D_{\varphi,v} : LW(v) \longrightarrow LW(v)$.

Next, we define the **local stable Whitehead graph** at v, denoted $SW(v)$. This is the subgraph of $LW(v)$ obtained by restricting to the periodic directions at v and the edges between them.

Now we define the **ideal Whitehead graph** of f, denoted $IW(f)$. This will take some preliminary definitions (all of which are explained in much greater detail in [HM08]). In this paper, we will only used ideal Whitehead graphs to find the indices of automorphisms, and the end of this section gives a practical way of constructing ideal Whitehead graphs.

Consider the Gromov boundary δF_n of the free group F_n. Then the action of $Aut(F_n)$ extends to an action on δF_n. For $F \in Aut(F_n)$, the extension is denoted by $\hat{F} : \delta F_n \longrightarrow \delta F_n$. Denote the set of nonrepelling fixed points of $\hat{F}$ by $Fix_N \hat{F}$. We call F a **principal automorphism** if $Fix_N \hat{F}$ contains at least three points. Let $f \in Out(F_n)$. The set of principal automorphisms representing f is denoted $PA(f)$, and $F, F' \in PA(f)$ are considered equivalent if there is an $x \in F_n$ such that $F' = i_x \circ F i_x^{-1}$, where i_x is the inner automorphism given by $i_x(y) = xyx^{-1}$. By Corollary 2.18 of [HM08], leaves of Λ can be seen as points $(P_1, P_2) \in D^2 \delta F_n$. For every principal automorphism F representing f, let $L(F)$ be the set of leaves (P_1, P_2) of Λ with $P_1, P_2 \in Fix_N(\hat{F})$. The **ideal Whitehead graph** has a component $W(F)$ given by the graph with one vertex for each point in $Fix_N(\hat{F})$, and an edge connecting $P_1, P_2 \in Fix_N(\hat{F})$ whenever there is a leaf $(P_1, P_2) \in \Lambda(F)$. For any $P \in \delta F_n$, there is at most one principal automorphism F such that $P \in Fix_N(\hat{(F)})$ (Corollary 2.9 in [HM08]). This means for $F' \neq F$, $L(F)\hat{L}(F') = \varnothing$. Thus it makes sense to define the ideal Whitehead graph as a disjoint union of components $W(F)$, each corresponding to a principal automorphism F representing f.

As explained in [HM08], here is a way of constructing the **ideal Whitehead graph** $IW(f)$ for nongeometric automorphisms. Let $\varphi : \Gamma \longrightarrow \Gamma$ be a train track representative of f. A **principal vertex** is a vertex with at least three fixed directions or one that is the endpoint of an NP. For each principal vertex v of Γ, we construct $SW(v)$. Then these graphs get glued together according to the two directions at the two ends of each NP in Γ (each such direction is a fixed direction at a principal vertex, hence a point in one of the $SW(v)$s). This way we get components of the $IW(f)$, each of which consists of stable Whitehead graphs glued together according to NPs. Furthermore, as proven in [HM08], the way the stable Whitehead graphs get glued, each NP gives a cut vertex in the ideal Whitehead graph.

The **index** of f is $i(f) = \sum_{C, \text{ component of } IW(g)} (1 - \frac{\#(\text{vertices of } C)}{2})$. The index has proven to be a useful way of distinguishing automorphisms and analyzing their properties.

One thing to note is that for a geometric or a parageometric automorphism f, $i(f) = 1 - n$, while for an ageometric automorphism of rank n, $0 \geq i \geq \frac{3}{2} - n$.

CHAPTER 2

PRELIMINARY PROOFS

In this chapter, using primarily results from [AK08], we show that there is a finite neighborhood of an axis of a hyperbolic irreducible automorphism g that contains the minset of any power of g. This allows us to focus on a finite number of g-orbits of simplices. Using this, we come up with a power of g that is rotationless and sends candidate loops to loops that are legal except for the illegal turns taken by Nielsen Paths they contain (see Section 1.2 for all the definitions). This is the power that will potentially have a minset that is stable under further powers.

We begin by showing that the minset of a hyperbolic irreducible automorphism is a bounded distance from its axis.

Lemma 2.1 *Let Γ be a graph and α a loop in Γ. Let g be a hyperbolic irreducible automorphism. Then $LEG_{g^k}(\alpha, \Gamma) \geq LEG_g(\alpha, \Gamma)$.*

Proof. All the terminology, definitions, and facts used are from [AK08].

$$LEG_{g^k}(\alpha, \Gamma) = \frac{\text{total length of legal subpaths in } \alpha \text{ of length } > \kappa_{g^k}}{l_\Gamma \alpha}$$

with $\kappa_{g^k} = \dfrac{4BCC(g^k)}{\lambda^k - 1}$, where $BCC(g^k)$ is the bounded cancellation constant. Note that from the definition of the bounded cancellation constant, $BCC(g^k) \leq \dfrac{\lambda^k - 1}{\lambda - 1} BCC(g)$, so

$$\kappa_{g^k} \leq 4\frac{\lambda^k - 1}{\lambda - 1} BCC(g) \frac{1}{\lambda^k - 1} = \frac{4BCC(g)}{\lambda - 1} = \kappa_g$$

This means (length of legal pieces in α of length $> \kappa_{g^k}$) $\geq$ (length of legal pieces in α of length $> \kappa_g$) so $LEG_{g^k}(\alpha, \Gamma) \geq LEG_g(\alpha, \Gamma)$. ∎

Lemma 2.2 *Let $\mathcal{L}_\varphi$ be an axis of g. Then the minset of any power of g is a bounded distance from $\mathcal{L}_\varphi$, and this bound can be chosen uniformly for all powers.*

Proof. Let $m \in \mathbb{N}$. Let λ be the stretch constant of g^m. Then $\mathcal{L}_\varphi$ is an axis for g^m as well. For $X \in \mathcal{X}_n$, let $\pi(X)$ be the projection of X to $\mathcal{L}_\varphi$, as mentioned in Section 1.2 and described fully in [AK08]. By Corollary 5.11 of [AK08], $\exists s, c > 0$ such that if $d(\pi(X), \pi(Y)) > s$, then

$$d(X,Y) > d(X, \pi(X)) + d(\pi(X), \pi(Y)) - c$$

Suppose $\Gamma \in M(g^m)$. Then $\exists k$ such that $d(\pi(\Gamma), \pi(g^{mk}(\Gamma))) = d(\pi(\Gamma), g^{mk}(\pi(\Gamma)) > s$. Then

$$d(\Gamma, g^{mk}(\Gamma)) \geq d(\Gamma, \pi(\Gamma)) + d(\pi(\Gamma), \pi(g^{mk}(\Gamma))) - c$$

Since $\Gamma \in M(g^m)$, $d(\Gamma, g^{mk}(\Gamma)) = k\lambda$, and $d(\pi(\Gamma), \pi(g^{mk}(\Gamma))) = d(\pi(\Gamma), g^{mk}(\pi(\Gamma)) = k\lambda$. Now the equation above becomes

$$k\lambda \geq d(\Gamma, \pi(\Gamma)) + k\lambda - c$$

so $d(\Gamma, \pi(\Gamma)) \leq c$.

It remains to address how these constants s and c change with powers of g to ensure that we can find a uniform bound on the distance. In Corollary 5.11 of [AK08], the constants s and c come from Proposition 5.7 and Proposition 5.8. The proofs of both propositions use constants generated by using the projections of graphs to the axis (which depend only on the axis and not on the power of the automorphism) and constants that ensure that $LEG_g(\alpha, \Gamma) > \epsilon$ for different choices of loop α in a graph Γ and constant ϵ. By Lemma 2.1, $\forall k \in \mathbb{N}$, $LEG_{g^k}(\alpha, \Gamma) \geq LEG_g(\alpha, \Gamma) > \epsilon$, so the constants s and c are nonincreasing with powers of g. ∎

Lemma 2.3 *Let g be a hyperbolic irreducible automorphism. There are only finitely many g-orbits of simplices that intersect the minset of any power of g.*

Proof. There are only finitely many g-orbits of simplices that are a bounded distance from $\mathcal{L}_\varphi$, hence there are only finitely many g-orbits that intersect the minset of any power of g.

∎

Lemma 2.4 *Let g be a hyperbolic irreducible automorphism. Let $\Gamma \in M(g)$. Let μ be a loop in Γ. There exists a $k \in \mathbb{N}$, such that $g^k_\Gamma(\mu)$ is legal except for the single illegal turn taken by each NP $g^k_\Gamma(\mu)$ contains (so legal if $g^k_\Gamma(\mu)$ does not contain any NPs). Call this the **special power** for μ in Γ.*

Proof. This is Proposition 3.1 in [BF94]. ∎

The next lemma shows that the special power in Lemma 2.4 can be chosen uniformly for all candidate loops in Γ.

Lemma 2.5 *Let g be a hyperbolic irreducible automorphism. Let $\Gamma \in M(g)$. There exists a $k \in \mathbb{N}$ such that for any candidate loop $\mu \in \Gamma$, $g_\Gamma^k(\mu)$ is legal except for the single illegal turn taken by each NP $g_\Gamma^k(\mu)$ contains (so legal if $g_\Gamma^k(\mu)$ does not contain any NPs). Any power of g that satisfies this condition, call a **special power** of g for Γ.*

Proof. See Section 1.2.1 for an explanation of candidate loops. There are finitely many candidate loops in Γ, so by taking the least common multiple of their special powers, we get a power that works for all of them.

∎

Now we can work toward showing that this special power can be chosen uniformly for all graphs in the minset of any power of g.

Definition 2.1 *Let g be a rotationless hyperbolic irreducible automorphism. Let S be a simplex such that $S \cap M(g) \neq \varnothing$. For each graph $\Gamma \in S \cap M(g)$ and for every NP in Γ, define the Nielsen Path Equivalence Class **NPE** of γ to be the set of all NPs γ' in graphs in $S \cap M(g)$ such that a loop contains γ if and only if it contains γ'.*

Lemma 2.6 *If Γ and Γ' in $S \cap M(g)$ have the same NPEs, they have the same minimal train track structure. Note: If two graphs do not have NPs, we consider them to have the same NPEs.*

Proof. Endow Γ and Γ' with the minimal train track structure w.r.t. g. Let t be a legal turn in Γ. Then there exists in Γ a legal loop μ and an $M \in \mathbb{N}$ such that $\forall s \geq M$, $g_S^s(\mu)$ crosses t. By Lemma 2.4, there is an $m \geq M$ such that in Γ', $g_S^m(\mu)$ is either legal or contains an NP. However, since μ is legal in Γ, $g_S^m(\mu)$ does not contain an NP in Γ. Since Γ and Γ' have the same NPEs, $g_S^m(\mu)$ does not contain an NP in Γ'. This means $g_S^m(\mu)$ is legal in Γ'. Now for every $s \in \mathbb{N}$, $g_S^s(g_S^m(\mu))$ crosses t, so we have a legal loop in Γ' every image of which under g_S crosses t, meaning t is legal in Γ'. We just showed that any turn that is legal in Γ is legal in Γ'. Reversing the roles of Γ and Γ', we can show the same implication in the opposite direction. So Γ and Γ' have the same train track structure.

∎

Proposition 2.7 *Let g be a hyperbolic irreducible automorphism. There exists a $k \in \mathbb{N}$ such that for any graph $\Gamma \in M(g^k)$ and any candidate loop $\mu \in \Gamma$, $g_\Gamma^k(\mu)$ is legal except for the single illegal turn taken by each NP $g_\Gamma^k(\mu)$ contains (so legal if $g_\Gamma^k(\mu)$ does not contain any NPs).*

Proof. Lemma 2.6 shows that all the graphs in $S \cap M(g)$ that have the same NPEs have the same train track structure. This means that the same special power (see Lemma 2.5) works for all these graphs. For a marked graph representing S, there are finitely many possible combinations of NPEs (note that this is just based on the loops in the graph and does not depend on the automorphism or its powers). By taking the composition of the special powers for each combination contained in graphs in $S \cap M(g)$, we get a power g^{k_1} that is a special power for every graph in $S \cap M(g)$. Since $S \cap M(g^{k_1})$ might be bigger than $S \cap M(g)$, we might get new NPE combinations. Repeat the process for those to get a power g^{k_2}. Keep doing this until all possible NPE combinations for graphs in $S \cap M(g^t)$ for any $t \in \mathbb{N}$ have been used, and we have a power of g, call it h, such that for every possible NPE combination, h is a special power for some graph in $S \cap M(h)$ with that combination.

Claim 2.8 *Suppose $\Gamma' \in S \cap M(h^k)$. Then h^k is a special power for Γ'.*

Proof. Let Γ be a graph in $S \cap M(h)$ for which h is a special power and that has the same NPEs as Γ'. Such a Γ exists by the way we constructed h in the previous paragraph. Let t be a legal turn in Γ. Then in Γ, there exists a legal loop μ and an $M \in \mathbb{N}$ such that $\forall s \geq M$, $h_S^s(\mu)$ crosses t. By Lemma 2.4, there is an $m \geq M$ such that in Γ', $h_S^{km}(\mu)$ is either legal or contains an NP. However, since μ is legal in Γ, $h_S^{km}(\mu)$ does not contain an NP in Γ. Since Γ and Γ' have the same NPEs, $h_S^{km}(\mu)$ does not contain an NP in Γ'. This means $h_S^{km}(\mu)$ is legal in Γ'. Now for every $s \in \mathbb{N}$, $h_S^{ks}(h_S^{km}(\mu))$ crosses t, so we have a legal loop in Γ' every image of which under h_S crosses t, meaning t is legal in Γ'. We just showed that any turn that is legal in Γ w.r.t. h is legal in Γ' w.r.t. h^k. Now, since the image of any loop under h^k is legal other than NPs in Γ, it is legal other than NPs in Γ'. This means h^k is a special power for Γ'.

∎

Since there are only finitely many g-orbits of simplices that intersect $M(g^k)$ for any k, we can take the product of their powers of g that give an h as above to get a power f that is a power of all the hs. Now for every $\Gamma \in M(f)$, f is a special power.

∎

CHAPTER 3

STABILITY OF MINSET

For a hyperbolic irreducible automorphism f, there is a natural inclusion $M(f) \subseteq M(f^k)$, as shown in [BFH97] . However, the minset might increase as we take powers of f. This chapter shows that the minset of a hyperbolic irreducible automorphism eventually stabilizes under further powers if and only if no train track representative of the automorphism has a Pre-Nielsen Path.

3.1 Pre-Nielsen Path Implies Never Stable Minset

The goal of this section is to prove the following theorem:

Theorem 3.1 *If a hyperbolic irreducible automorphism f has a train track representative Γ that has a PNP, then $M(f)$ does not stabilize under further powers of any power of f, i.e., $\forall k \in \mathbb{N}\ \exists m \in \mathbb{N}$ such that $M(f^k) \subsetneq M(f^{mk})$.*

First, we show that if we have an arbitrary PNP, i.e., a path that maps to an NP and is not itself an NP, we can use it to find a PNP that gives an unstable minset.

Lemma 3.1 *Let f' be a hyperbolic irreducible automorphism with a train track representative Γ' that has a PNP. Take a power, if necessary, so that f' is rotationless and such that exists a train track map $\varphi' : \Gamma' \longrightarrow \Gamma'$ with a PNP γ' such that $\varphi'(\gamma') = \alpha'\bar{\beta}'$ is an NP. Then there is a power f of f' and a graph Γ_0 in $M(f)$ that has a PNP γ and NP $\alpha\bar{\beta}$ s.t. γ has exactly one illegal turn and this turn is distinct from the illegal turn of $\alpha\bar{\beta}$.*

Proof. First, we address the case when γ' has illegal turns that coincide with the illegal turn of $\alpha'\bar{\beta}'$. Pick one such turn t_1. Then we can separate α' into two paths α'_1 and α'_2 and β' into two paths β'_1 and β'_2 s.t. $\alpha'_2\bar{\beta}'_2$ contains the illegal turn of $\alpha'\bar{\beta}'$, there is a subpath of γ' contains $\alpha'_2\bar{\beta}'_2$ that contains t_1, and γ' does not contain the end segments of α'_1 and β'_1 in a neighborhood of $\alpha'_2\bar{\beta}'_2$. Note that either α'_1 or β'_1 must have nonzero length. Assume without loss of generality that $\alpha'_1 > \beta'_1$. Split β'_2 into a path β'_4 of the same length as α'_2 and a path β'_3. Let γ'_1 be the part of γ' connecting to α'_2 and γ'_2 the part connecting to β'_2. Figure 3.1 illustrates this. Now there is a power of f', call it f, such that $f(\alpha'_2) = f(\beta'_4)$. This means we can fold the illegal turn of $\alpha'_2\bar{\beta}'_4$ for the length α'_2 and get a train track representative $\varphi'' : \Gamma'' \longrightarrow \Gamma''$. Let α'' be the image of α'_1 after the fold, β'' be the image of $\beta'_1\beta'_3$, and γ'' be the image of $\gamma'_1\bar{\beta}'_3\gamma'_2$. Then γ'' is a PNP mapping to the NP $\alpha''\bar{\beta}''$, and the image after the fold of the turn t_1 is distinct from illegal turn of $\alpha''\bar{\beta}''$. If γ'' has an illegal turn coinciding with the llegal turn of $\alpha''\bar{\beta}''$, repeat this process to get a new PNP and NP. Since γ' does not get any new illegal turns while we fold, and γ' start with a finite number of illegal turns, we only need to repeat this process finitely many times to get a final PNP γ'' mapping to the NP $\alpha''\bar{\beta}''$ that does not contain the illegal turn of $\alpha''\bar{\beta}''$.

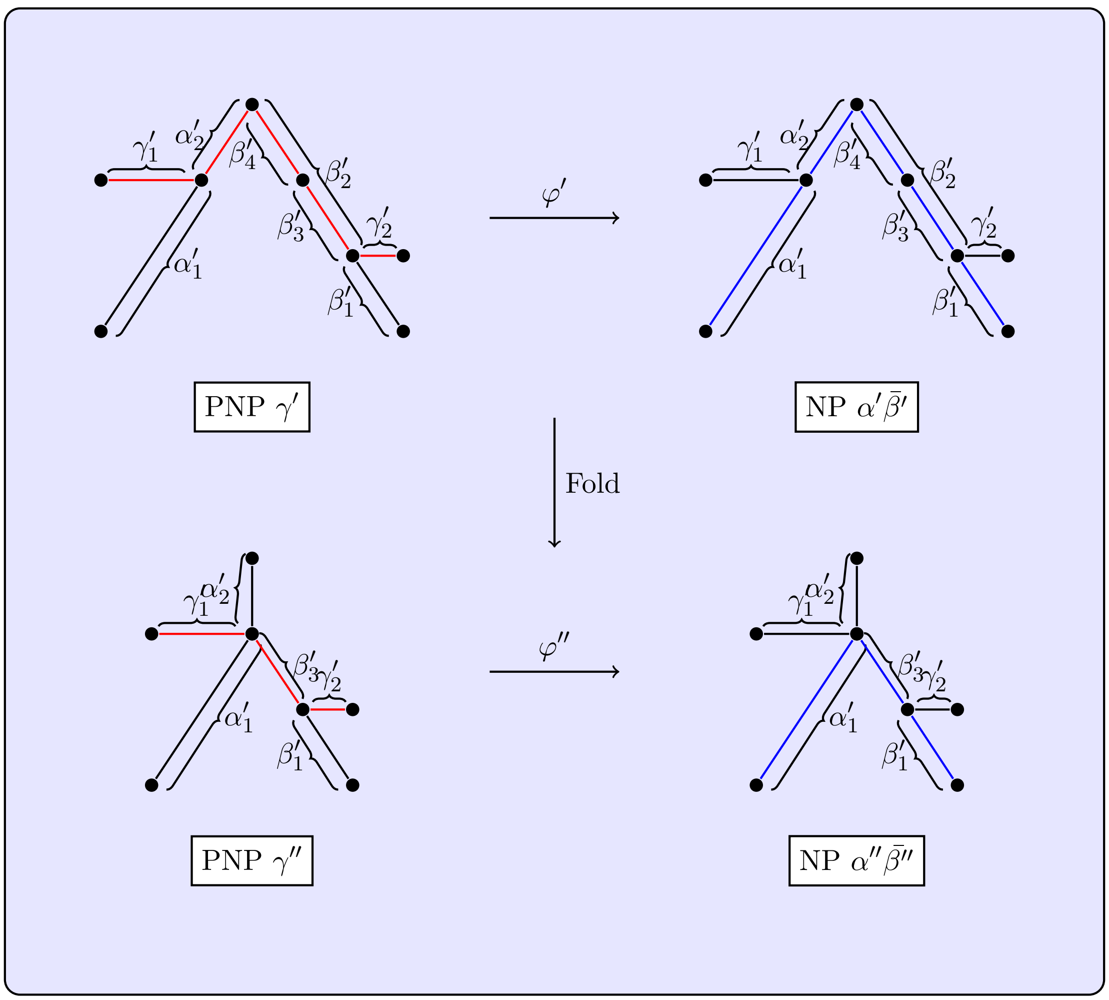

Figure 3.1. Pre-Nielsen Path and Nielsen Path with same illegal turn

If the illegal turn of γ' is already distinct from the illegal turn of $\alpha'\bar{\beta}'$, then simply relabel f', φ', α', β', and γ' by f, φ'', α'', β'', and γ'', respectively.

Let λ be the stretch constant of f. Let c'' be the cancellation under φ'' of γ'', i.e., $c'' = \lambda l''_{\Gamma}(\gamma'') - l''_{\Gamma}(\alpha''\bar{\beta}'')$ (c'' might be zero). Fold any illegal turns in γ'' for a total length $\frac{c''}{2}$ to get a new graph Γ_0 without any cancellation under the first power $\varphi : \Gamma_0 \longrightarrow \Gamma_0$, as shown in Figure 3.2.

Let γ and $\alpha\bar{\beta}$ be the images of γ'' and $\alpha''\bar{\beta}''$ after the fold. Now in Γ_0, γ is a PNP mapping to the NP $\alpha\bar{\beta}$ with no cancellation and s.t. γ does not contain the illegal turn of $\alpha\bar{\beta}$. This means γ has a single illegal turn that is distinct from the illegal turn of $\alpha\bar{\beta}$.

∎

We can fold the single illegal turn in γ to obtain a family of graphs Γ_x, where $0 \leq x < \frac{l_{\Gamma_0}(\gamma)}{2}$ is the length we fold (Figure 3.3). Note that as explained in Section 1.2, as we fold, we are potentially leaving $M(f)$, but the map is still defined in terms of tightened images of loops, and these are the same as for Γ_0. For a loop μ, this image is $f_{\Gamma_x}(\mu)$.

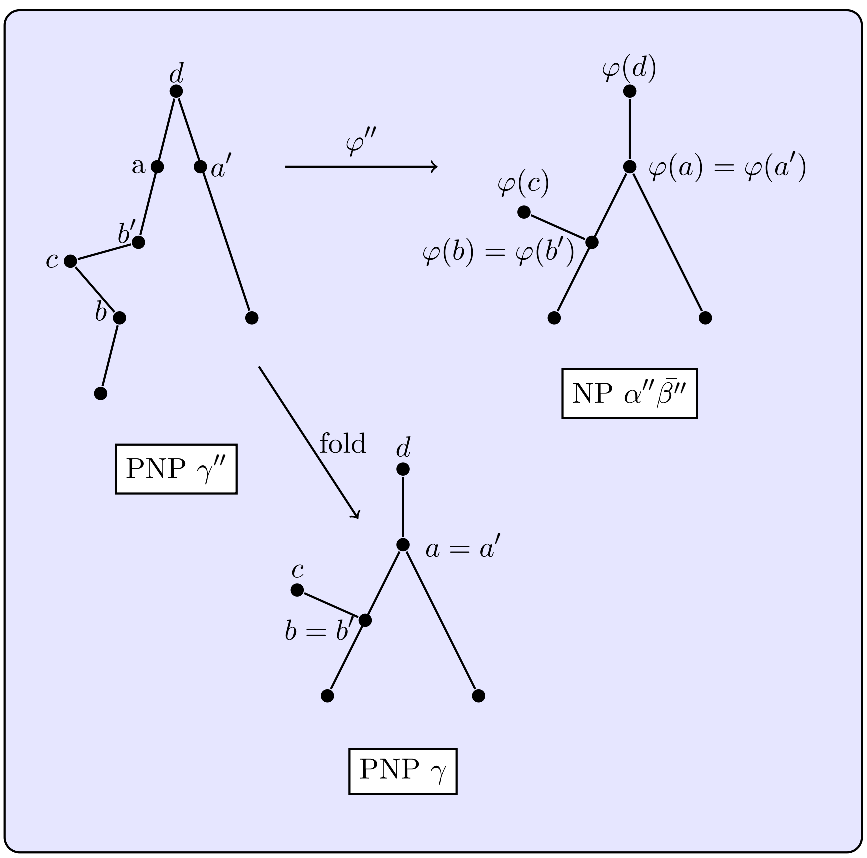

Figure 3.2. Cancellation in γ''

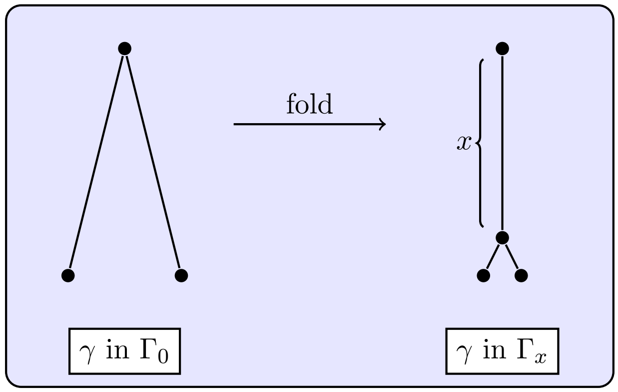

Figure 3.3. Folding γ to get Γ_x

Our goal is to show that as we take powers, we are able to fold longer segments without leaving the minset, hence we have a minset that never stabilizes.

Now we prove several lemmas with the setup described above. Consider lengths in Γ_x as if we fold without rescaling. We ultimately only require the stretch of loops, which is a ratio of lengths equally affected by rescaling, so we can just disregard rescaling to make the computations easier.

Lemma 3.2 *Suppose f' is a rotationless hyperbolic irreducible automorphism that has a representative with a PNP. As described in Lemma 3.1, there is a power of f', call it f, and a train track representative Γ_0 of f with a PNP γ that maps to an NP $\alpha\bar{\beta}$ s.t. γ has one illegal turn, which is distinct from the illegal turn of $\alpha\bar{\beta}$, and Γ_x is obtained by folding the illegal turn in γ for a distance x. Let ν be a loop that does not cross the illegal turn of γ. Then $l_{\Gamma_x}(\nu) = l_{\Gamma_0}(\nu)$.*

Proof. The proof consists of three cases.

Case 1 If a path ρ does not contain any part of γ, it is not affected by the fold.

Case 2 If a path ρ enters one side of γ and leaves, there is no cancellation, so its length is not affected by the fold (Figure 3.4).

Case 3 Suppose both sides of γ contain the same vertex v, and γ enters one side and jumps to the other at v.

Claim 3.3 *Let y_1 and y_2 be the distances from the illegal turn of γ to v along the two sides of γ. Then $y_1 \neq y_2$.*

Proof. Since there is no cancellation in γ, the distances from the illegal turn of $\alpha\bar{\beta}$ to $f_{\Gamma_0}(v)$ along α and β are λy_1 and λy_2. Suppose $y_1 = y_2$. Then there is a power m of f

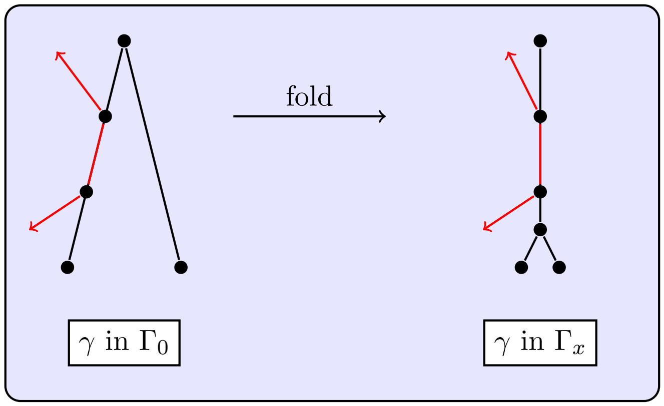

Figure 3.4. Path through one side of γ.

such that φ^m identifies the segments connecting the illegal turn of $\alpha\bar{\beta}$ to $\varphi(v)$ in α and in β (Figure 3.5). However, then we have a nontrivial loop consisting of these two segments mapping to a trivial loop, contradicting the fact that f^m is an automorphism.

■

Now, as we fold γ, the length of the path is not affected. Figure 3.6 demonstrates just one way the path can enter and leave γ, but since by Claim 3.3, the two copies of v will not coincide as we fold, there will not be any cancellation in the path.

■

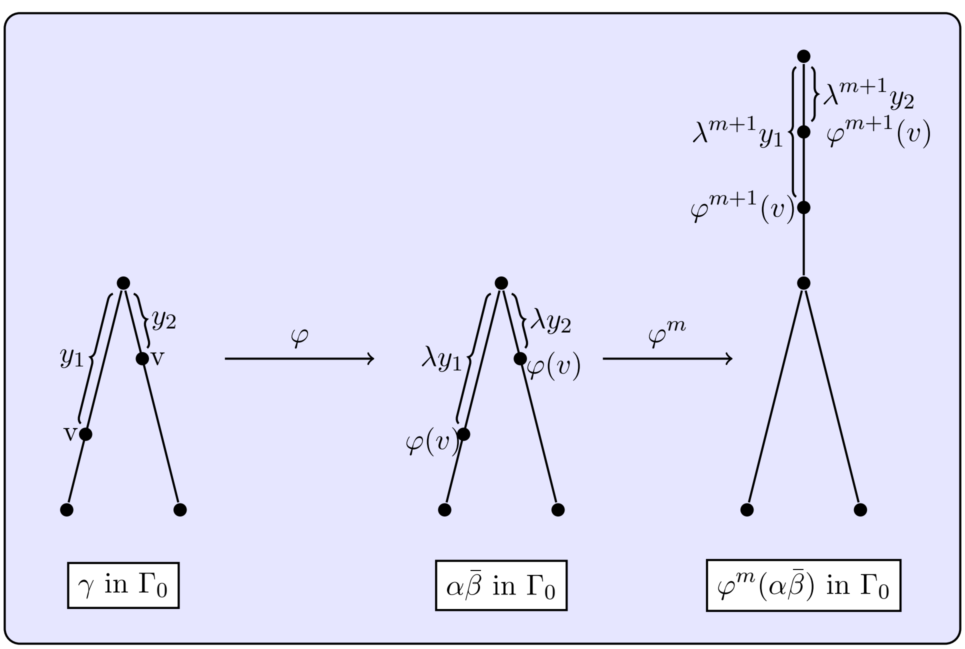

Figure 3.5. Same vertex on different sides of γ.

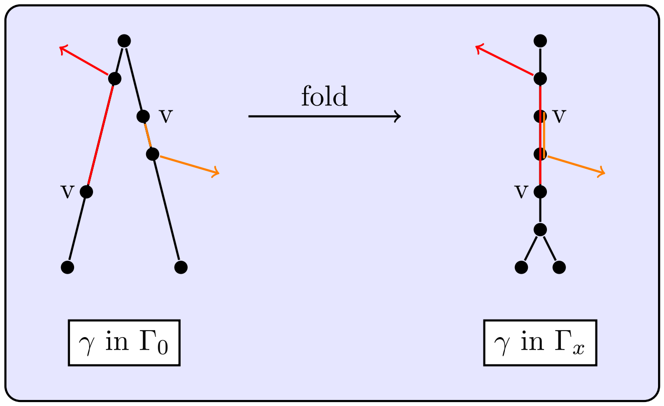

Figure 3.6. Path through both sides of γ.

Lemma 3.4 *Suppose f' is a rotationless hyperbolic irreducible automorphism that has a representative with a PNP. As described in Lemma 3.1, there is a power of f', call it f, and a train track representative Γ_0 of f with a PNP γ that maps to an NP $\alpha\bar{\beta}$ s.t. γ has one illegal turn, which is distinct from the illegal turn of $\alpha\bar{\beta}$, and Γ_x is obtained by folding the illegal turn in γ for a distance x. Let λ be the stretch constant of f. Let c be the cancellation under φ of $\alpha\bar{\beta}$, i.e., $c = \lambda l_\Gamma(\alpha\bar{\beta}) - l_\Gamma(\varphi(\alpha\bar{\beta}))$ ($c > 0$). We show this in Figure 3.7. Let μ be a loop in Γ_x that contains γ and is legal other than the single illegal turn in γ. This situation is illustrated in Figure 3.8. Let $d = l_{\Gamma_0}(\mu) - l_{\Gamma_0}(\gamma)$. The stretch under f^m on μ is equal to $\dfrac{\lambda^m d - \lambda l_{\Gamma_0}(\gamma)}{d + l_{\Gamma_0}(\gamma) - 2x}$.*

Proof. Since we folded γ to get to Γ_x, $l_{\Gamma_x}(\mu) = d + l_{\Gamma_0}(\gamma) - 2x$, while $l_{\Gamma_x}(\alpha) = l_{\Gamma_0}(\alpha) = \dfrac{\lambda l_{\Gamma_0}(\gamma)}{2}$. Let C_s be the total cancellation of $\alpha\bar{\beta}$ after applying $f^s_{\Gamma_x}$ (Figure 3.8).

As above, let c be the cancellation of $\alpha\bar{\beta}$ after applying f. Then

$$C_s = c + \lambda c + \lambda^2 c + ... + \lambda^{s-1} c = \frac{\lambda^s - 1}{\lambda - 1} c$$

Since $c = 2(\lambda l_{\Gamma_0}(\alpha) - l_{\Gamma_0}(\alpha)) = 2 l_{\Gamma_0}(\alpha)(\lambda - 1)$,

$$C_s = 2 l_{\Gamma_0}(\alpha)(\lambda - 1)\frac{\lambda^s - 1}{\lambda - 1} = 2 l_{\Gamma_0}(\alpha)(\lambda^s - 1) = \lambda l_{\Gamma_0}(\gamma)(\lambda^s - 1)$$

Now $l_{\Gamma_x}(f_{\Gamma_x}(\mu)) = l_{\Gamma_0}(f_{\Gamma_x}(\mu)) = \lambda^m d + \lambda^m l_{\Gamma_0}(\gamma) - C_{m-1} = \lambda^m d + \lambda l_{\Gamma_0}(\gamma)$.

This means

$$\sigma_{\Gamma_x, f^m}(\mu) = \frac{l_{\Gamma_x}(f(\mu))}{l_{\Gamma_x}(\mu)} = \frac{\lambda^m d - \lambda l_{\Gamma_0}(\gamma)}{d + l_{\Gamma_0}(\gamma) - 2x}$$

.

■

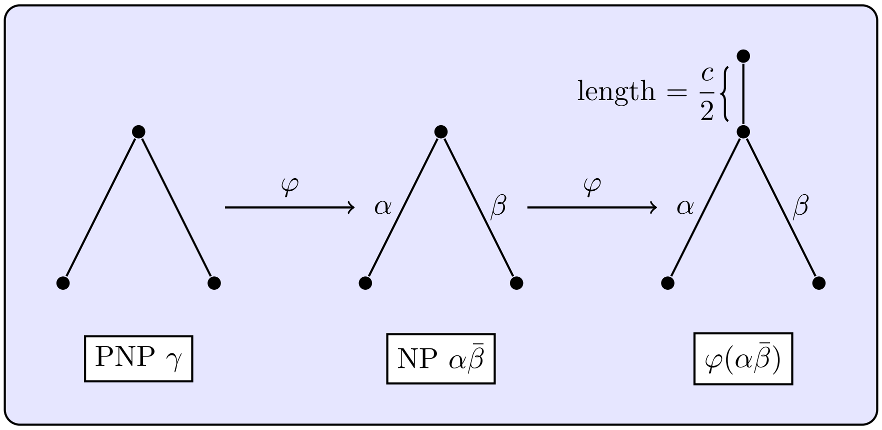

Figure 3.7. The NP and PNP in Γ_0

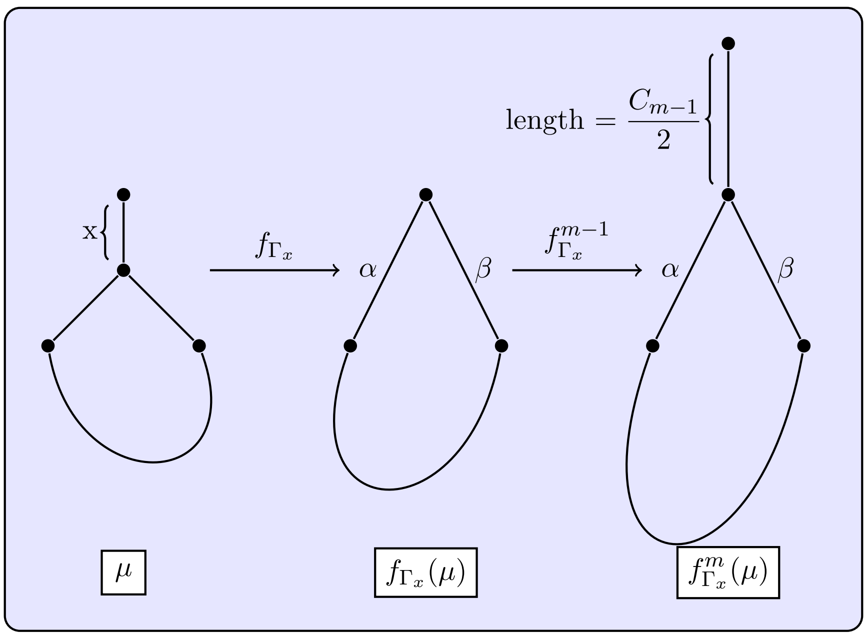

Figure 3.8. Legal loop containing γ in Γ_x

Lemma 3.5 *Whenever* $x \leq \frac{l_{\Gamma_0}(\gamma)}{2}(1 - \frac{1}{\lambda^{m-1}})$, *we have* $\sigma_{\Gamma_x, f^m}(\mu) \leq \lambda^m$. *This does not depend on* $l_{\Gamma_0}(\mu)$, *so this holds for any loop that contains* γ *and is legal everywhere else.*

Proof. From Lemma 3.4 we get $\sigma_{\Gamma_x, f^m}(\mu) \leq \lambda^m$ if and only if $\frac{\lambda^m d - \lambda l_{\Gamma_0}(\gamma)}{d + l_{\Gamma_0}(\gamma) - 2x} \leq \lambda^m$ if and only if

$$x \leq \frac{\lambda^m l_{\Gamma_0}(\gamma) - \lambda l_{\Gamma_0}(\gamma)}{2\lambda^m} = \frac{l_{\Gamma_0}(\gamma)}{2}(1 - \frac{1}{\lambda^{m-1}})$$

.

∎

Lemma 3.6 *Let μ be a loop containing γ, but with possible other illegal turns. When $x \leq \frac{l_{\Gamma_0}(\gamma)}{2}(1 - \frac{1}{\lambda^{m-1}})$, we have $\sigma_{\Gamma_x, f^m}(\mu) \leq \lambda^m$.*

Proof. Following the proof of Lemma 3.4, we get

$$\sigma_{\Gamma_x, f^m}(\mu) \leq \frac{\lambda^m(l_{\Gamma_0}(\mu) - l_{\Gamma_0}(\gamma)) - \lambda l_{\Gamma_0}(\gamma)}{(l_{\Gamma_0}(\mu) - l_{\Gamma_0}(\gamma)) + l_{\Gamma_0}(\gamma) - 2x}$$

This means, like in the proof of Lemma 3.5 but with the implication in just one direction, if $x \leq \frac{l_{\Gamma_0}(\gamma)}{2}(1 - \frac{1}{\lambda^{m-1}})$, then $\sigma_{\Gamma_x, f^m}(\mu) \leq \lambda^m$.

∎

Lemma 3.7 *The stretch under f^m on any loop μ in Γ_x that does not contain the entire PNP γ is less than or equal to λ^m for high enough m, i.e., $\exists k \in \mathbb{N}$, such that $\forall x$ and $m \geq k$, $\sigma_{\Gamma_x, f^m}(\mu) \leq \lambda^m$.*

Proof. If μ does not cross the illegal turn in Γ, then by Lemma 3.2, for any $m \in \mathbb{N}$, $\sigma_{\Gamma_x, f^m}(\mu) = \sigma_{\Gamma_0, f^m}(\mu) \leq \lambda^m$.

Now suppose μ crosses the illegal turn of γ but does not contain the entire path. Let ρ be the the shorter of the two legal subpaths of γ contained in μ. Let $\varphi : \Gamma_x \longrightarrow \Gamma_x$ be an optimal map. Without loss of generality, assume $\varphi(\rho) \subseteq \alpha$. Then $\exists k \in \mathbb{N}$ such that C_{k-1} (as in the proof of Lemma 3.4) is greater than $2l_{\Gamma_0}(\varphi^k(\rho))$. Then for any $m \geq k$, we have $\varphi^m(\rho) \cap \alpha = \varnothing$. Then $\forall x$, $\sigma_{\Gamma_x, f^m}(\mu) = \sigma_{\Gamma_0, f^m}(\mu) \leq \lambda^m$.

∎

Now we can prove the main theorem of this section.

Proof.[Proof of Theorem 3.1] Suppose a hyperbolic irreducible automorphism f' has a train track representative that has a PNP. Take a power of f' to get a rotationless automorphism. As described in Lemma 3.1, there is a power of f', call it f, and a train track representative Γ_0 of f with a PNP γ that maps to an NP $\alpha\bar{\beta}$ s.t. γ has one illegal turn, which is distinct from the illegal turn of $\alpha\bar{\beta}$, and Γ_x is obtained by folding the illegal turn in γ for a distance x. Let λ be the stretch constant of f.

Let μ be a loop containing γ and legal everywhere else. Then by Lemma 3.5, whenever $x \leq \frac{l_{\Gamma_0}(\gamma)}{2}(1 - \frac{1}{\lambda^{m-1}})$, we have $\sigma_{\Gamma_x, f^m}(\mu) \leq \lambda^m$.

Let μ be a loop containing γ, but with possible other illegal turns. By Lemma 3.6, when $x \leq \frac{l_{\Gamma_0}(\gamma)}{2}(1 - \frac{1}{\lambda^{m-1}})$, we have $\sigma_{\Gamma_x, f^m}(\mu) \leq \lambda^m$.

Note that k in Lemma 3.7 depends only on $l_{\Gamma_0}(\rho)$. Since $l_{\Gamma_0}(\rho) \leq \frac{l_{\Gamma_0}(\gamma)}{2}-$(length of shortest edge of Γ_0), there is a universal k that works for all such loops. Fix such a k. Let μ be a loop in Γ_0 that does not contain the entire PNP γ. Then by Lemma 3.7, $\forall x$ and $m \geq k$, $\sigma_{\Gamma_x, f^m}(\mu) \leq \lambda^m$.

This covers all the possible loops. Since $\Gamma_x \in M(f^m)$ whenever the stretch on every loop in Γ_x is less than or equal to λ^m, we get that for $m \geq k$, $\Gamma_x \in M(f^m)$ if and only if

$$x \leq \frac{l_{\Gamma_0}(\gamma)}{2}(1 - \frac{1}{\lambda^{m-1}})$$

.

Since $\frac{l_{\Gamma_0}(\gamma)}{2}(1 - \frac{1}{\lambda^{m-1}}) \longrightarrow \frac{l_{\Gamma_0}(\gamma)}{2}$ as $m \longrightarrow \infty$ and

$$\frac{l_{\Gamma_0}(\gamma)}{2}(1 - \frac{1}{\lambda^{m'-1}}) > \frac{l_{\Gamma_0}(\gamma)}{2}(1 - \frac{1}{\lambda^{m-1}})$$

for $m' > m$, the minset never stabilizes.

Note that we have shown that for $m = 1$, Γ_x is in the minset only when $x = 0$. However, as $m \longrightarrow \infty$, $\frac{l_{\Gamma_0}(\gamma)}{2}(1 - \frac{1}{\lambda^{m-1}}) \longrightarrow \frac{l_{\Gamma_0}(\gamma)}{2}$, so we fold more and more of the PNP, approaching the length of the whole PNP, but never actually folding the whole thing.

∎

Now we give a specific example to explicitly demonstrate the process and results above. Throughout, we will use decimal approximations, because the exact forms involve numerous roots, so they are more complicated and less illustrative.

Let $n = 4$. Let f be the automorphism given by:

$$\begin{array}{ccc} a & \longrightarrow & aad \\ b & \longrightarrow & bd \\ c & \longrightarrow & a \\ d & \longrightarrow & cb \end{array}$$

Then f is a hyperbolic irreducible automorphism with $\lambda = 2.29$.

Let Γ_0 be R_4 with edges marked by a, b, c, and d, labeled with e_a, e_b, e_c, and e_d, respectively. Let the edge lengths be $l_{\Gamma_0}(e_a) = .513$, $l_{\Gamma_0}(e_b) = .115$, $l_{\Gamma_0}(e_c) = .224$, and $l_{\Gamma_0}(e_d) = .148$. Then Γ_0 is in $M(f)$.

The marked metric graph Γ_0 has one NP $\dot{a}B$, where the dot over the a means we are using just part of the edge e_a (specifically a segment of length .115). The NP has cancellation $c = .148$. This means cancellation under f^s of the NP is $C_s = \frac{2.29^s - 1}{1.19}$. There is also a PNP γ given by $\dot{e_c}\dot{e_D}$ (the segments in c and D that constitute the PNP are of length 0.05). Then $l_{\Gamma_0}(\gamma) = 0.10$. We show a picture of Γ_0 with the NP and PNP in Figure 3.9.

Now, as described in Section 3.1, for $m \in \mathbb{N}$, we can fold the turn for a length of $x_m = \frac{l_{\Gamma_0}(\gamma)}{2}(1 - \frac{1}{\lambda^{m-1}}) = .05(1 - (0.44)^{m-1})$ to get a graph Γ_{x_m} (Figure 3.10). Now $\Gamma_{x_m} \in M(f^m)$, but if we fold the turn $\{e_c, \bar{e_d}\}$ any further, we leave $M(f^m)$.

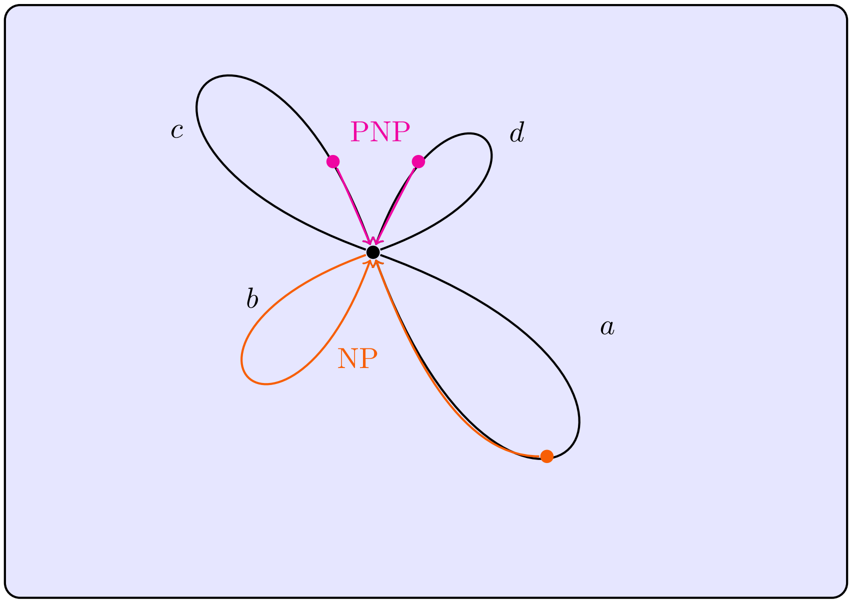

Figure 3.9. The graph Γ_0 with NP $\dot{a}B$ and PNP $\dot{c}\dot{D}$.

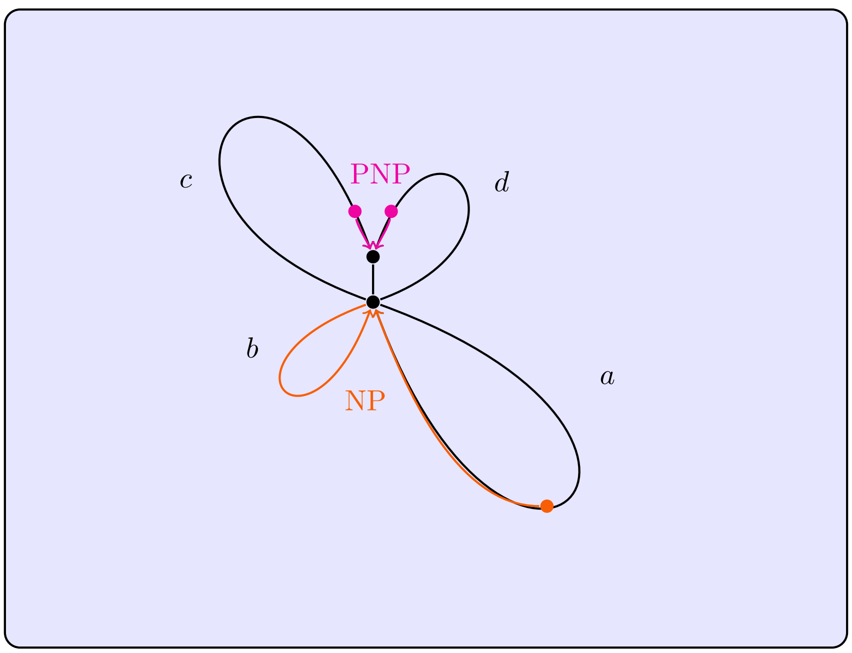

Figure 3.10. The graph Γ_{x_m} obtained by folding the illegal turn of the PNP.

As $m \longrightarrow \infty$, we can fold the turn $\{e_c, \bar{e_d}\}$ further and further, with $\frac{l_{\Gamma_0}(\gamma)}{2} = 0.05$ being the limit of the fold lengths, which is never achieved. This means, along the fold line, $M(f^m)$ increases as m increases and never stabilizes.

To illustrate this, we will show $M(f^m)$ in the 2-dimensional simplex in which we are folding. Let S be the simplex with missing faces consisting of graphs Γ as in Figure 3.11, with $l_\Gamma(e_a) = .513$, $l_\Gamma(e_b) = .115$, and $l_\Gamma(e_c) + l_\Gamma(e_d) + l_\Gamma(e_x) = .372$. In Figure 3.12, we show the simplex S and $M(f^m)$ in S to show a portion of the minset that does not stabilize.

3.2 No Nielsen Paths Implies Stable Minset

The goal of this section is the following theorem:

Theorem 3.2 *If no train track representative of a hyperbolic irreducible automorphism g has an NP, then $\exists k \in \mathbb{N}$ such that $M(g^k)$ is stable under further powers, i.e., $M(g^{mk}) = M(g^k)$ $\forall m \in \mathbb{N}$.*

Let g be a hyperbolic irreducible automorphism such that no representative of g has an NP.

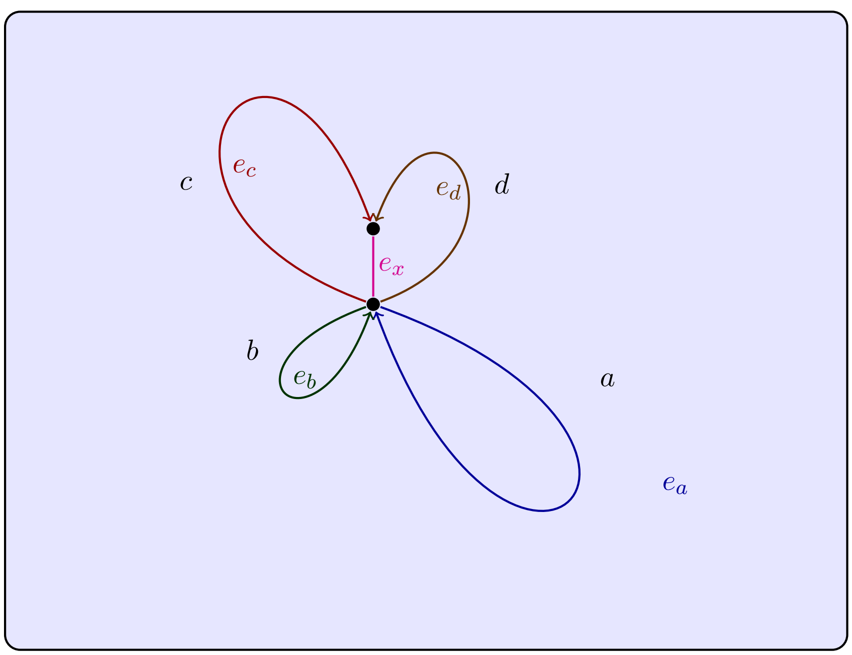

Figure 3.11. Γ in simplex S

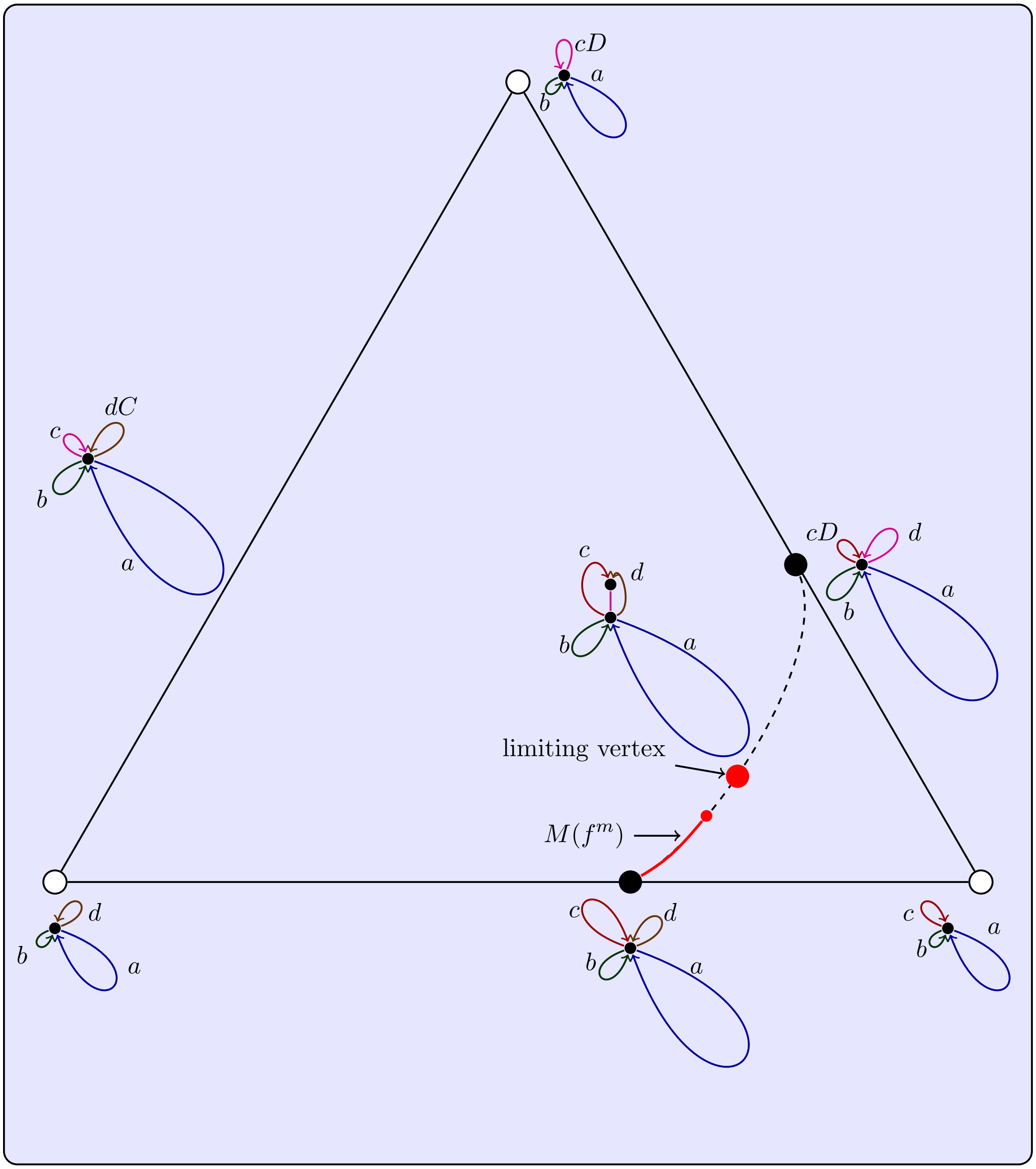

Figure 3.12. Simplex S containing never stable minset.

Let f be a power of g that is rotationless and such that for any graph $\Gamma \in M(f)$ and any candidate loop $\mu \in \Gamma$, $f_\Gamma(\mu)$ is legal (as found in Proposition 2.7).

Let $\Gamma \in M(f)$. Let λ be the stretch constant of f. Suppose $\Gamma' \in M(f^k)$ for some $k \in \mathbb{N}$ is in the same simplex S as Γ. Our goal is to show that $\Gamma' \in M(f)$.

Lemma 3.8 *Endow Γ and Γ' with the minimal train track structure w.r.t. f and f^k, respectively. Then if a loop μ is legal in Γ, it is legal in Γ'.*

Proof. Suppose t is a legal turn in Γ. Then $\exists \nu$, a legal loop in Γ, such that $\exists M \in \mathbb{N}$ such that $f_S^m(\nu)$ crosses t $\forall m \geq M$. Since there are no NPs, by Lemma 2.4, $\exists s \in \mathbb{N}$ such that $(f_S^k)^s(\nu)$ is legal in Γ' and such that $ks > M$. Take $m \in \mathbb{N}$. Then $(f_S^k)^m(f_S^{ks}(\nu)) = f_S^{km+ks}(\nu)$, so it crosses t. This means we have a legal loop $(f_S^{ks}(\nu))$ in Γ' the image of which under any power of f_S^k crosses t. Hence t is a legal turn in Γ'. This means that if a loop μ is legal in Γ, it is legal in Γ'.

∎

Lemma 3.9 *Let λ be the stretch constant of f. Let μ be a candidate loop in Γ' maximally stretched by f_S. Suppose $\sigma_{\Gamma',f}(\mu) = d$. Then $d = \lambda$*

Proof. In Γ, $f_S(\mu)$ is legal, because of the way we constructed f. Then by Lemma 3.8, $f_S(\mu)$ is legal w.r.t. f^k in Γ'. Thus $\forall m \in \mathbb{N}$, we have $\sigma_{\Gamma',f^{mk}}(f_S(\mu)) = \lambda^{mk}$. This means $\sigma_{\Gamma',f^{mk+1}}(\mu) = \lambda^{mk} d$

Let $L_m = f_S^{mk}(\mu)$.

Claim 3.10 *In Γ, the stretch by f_S on L_m approaches λ, i.e., $l_\Gamma(f_S(L_m)) \to \lambda l_\Gamma(L_m)$ as $m \to \infty$.*

Proof. Consider the lamination Λ corresponding to the train track on Γ. Then, since $f_S(\mu)$ is legal in Γ and $L_m = f_S^{m-1}(f_S(\mu))$, L_m is a leaf segment, and $l_\Gamma(L_m) \to \infty$ as $m \to \infty$. Now by Lemma 2.7 in [BFH97], $l_\Gamma(f_S(L_m)) \to \lambda l_\Gamma(L_m)$ as $m \to \infty$.

∎

Claim 3.11 *As $m \to \infty$, $l_{\Gamma'}(f_S(L_m)) \to \lambda l_{\Gamma'}(L_m)$.*

Proof. As explained in the proof of Lemma 2.7 in [BFH97], in a sufficiently long leaf L of Λ, edges appear with a definite frequency. For an edge $e \in \Gamma$, call this frequency F_e, so that $N_e(L) \sim F_e l_\Gamma(L)$, where $N_e(L)$ is the number of times the edge e appears in L.

Now, for large enough m, $N_e(f_S(L_m)) \sim l_\Gamma(f_S(L_m))F_e \sim \lambda l_\Gamma(L_m)F_e \sim \lambda N_e(L_m)$.

Since the only difference between Γ and Γ' is the metric, and in Γ and Γ' as marked graphs, the loops L_m and $f_S(L_m)$ are the same, so Γ' has the same $N_e(L_m)$ and $N_e(f_S(L_m))$. Then $l_{\Gamma'}(f_S(L_m)) = \sum_{e \in E(\Gamma')} N_e(f_S(L_m))l_{\Gamma'}(e) \sim \sum_{e \in E(\Gamma')} \lambda N_e(L_m)l_{\Gamma'}(e) = \lambda \sum_{e \in E(\Gamma')} N_e(L_m)l_{\Gamma'}(e) = \lambda l_{\Gamma'}(L_m)$.

■

This means $\sigma_{\Gamma', f^{mk+1}}(\mu) \to \lambda \sigma_{\Gamma', f^{mk}}(\mu)$ as $m \to \infty$. Since $\Gamma' \in M(f^k)$, $\sigma_{\Gamma', f^{mk}}(\mu) \leq \lambda^{mk}$, so $\forall \epsilon > 0$, there is an m such that $\sigma_{\Gamma', f_S^{mk+1}}(\mu) < \lambda^{mk+1} + \epsilon$.

From the beginning of the proof, we have $\sigma_{\Gamma', f^{mk+1}}(\mu) = \lambda^{mk} d$. Combining this with the previous inequality, we get $d \leq \lambda$, so $d = \lambda$. Figure 3.13 shows the commutative diagram that is the essence of this proof.

■

The Lemmas give the following proposition:

Proposition 3.12 *Let $\Gamma \in M(f)$. Suppose $\Gamma' \in M(f^k)$ for some $k \in \mathbb{N}$ is in the same simplex S as Γ. Then $\Gamma' \in M(f)$.*

Proof. From Lemma 3.9, we know that the stretch on any candidate loop is less than or equal to λ. By the definition of displacement as the maximal stretch on any candidate loop, it follows that $\Gamma' \in M(f)$. ■

We can now prove the Theorem:

Proof.[Proof of Theorem 3.1] For every simplex S such that $\exists k$ with $M(g^k) \cap S \neq \varnothing$, let k_S be the smallest power of g such that g^{k_S} is a special power as found in Proposition 2.7 and $M(g^{k_S}) \cap S \neq \varnothing$. Then by proposition 3.12, for any $m \in \mathbb{N}$, $M(g^{mk_S}) = M(g^{k_S})$.

$$\begin{tikzcd}
\mu \arrow[r, "d"] \arrow[d, "\leq \lambda^{mk}"'] & f_S(\mu) \arrow[d, "\lambda^{mk}"] \\
f_S^{mk}(\mu) \arrow[r, "\sim \lambda"] & f_S^{mk+1}(\mu)
\end{tikzcd}$$

Figure 3.13. Commutative diagram for no NP

Since there are finitely many g-orbits of simplices that intersect the minset of any power of g, we can take the product of their k_S's to get a power $g^{\prod_S k_S}$ such that $\forall m \in \mathbb{N}$, $M((g^{\prod_S k_S})^m) = M(g^{\prod_S k_S})$.

∎

3.3 Nielsen Paths but No Pre-Nielsen Paths Implies Stable Minset

The goal of this section is to prove the following theorem:

Theorem 3.3 *If no train track representative of a hyperbolic irreducible automorphism g has a PNP, then $\exists k \in \mathbb{N}$ such that $M(g^k)$ is stable under further powers, i.e., $M(g^{mk}) = M(g^k)$ $\forall m \in \mathbb{N}$.*

Let g be a hyperbolic irreducible automorphism such that no train track representative of f has a PNP. Let f be a power of g that is rotationless and such that for any graph $\Gamma \in M(f)$ and any candidate loop $\mu \in \Gamma$, $f_\Gamma(\mu)$ is legal other than illegal turns taken by Nielsen Paths (as found in Proposition 2.7). Let λ be the stretch constant of f.

Recall the definition of NPEs from Chapter 2.

Definition 3.4 *Let g be a rotationless hyperbolic irreducible automorphism. Let S be a simplex such that $S \cap M(g) \neq \varnothing$. For each graph $\Gamma \in S \cap M(g)$ and for every NP in Γ, define the Nielsen Path Equivalence Class* ***NPE*** *of γ to be the set of all NPs γ' in graphs in $S \cap M(g)$ such that a loop contains γ if and only if it contains γ'.*

We refine the definition by introducing candidate NPEs.

Definition 3.5 *An NPE is a* ***candidate NPE*** *if one of its representatives is contained in a candidate loop. Note that this means that every representative is contained in a candidate loop.*

Note that for an automorphisms with no PNPs, the image of a candidate loop under any power of the automorphism only contains candidate NPEs.

Lemma 3.13 *Let g be a rotationless hyperbolic irreducible automorphism. Let S be a simplex such that $S \cap M(g) \neq \varnothing$. Then the set of candidate NPEs in S is finite.*

Proof. Any representative of a candidate NPE x is contained in a candidate loop. Since a candidate loop crosses each edge at most twice, any representative crosses each edge at most twice. Consider the set T of loops that cross every edge of a graph representing S at most twice. Then, since every representative of x crosses every edge at most twice, x is determined by S. Since S is a finite set, the set of all possible candidate NPEs is finite.

■

Suppose $\Gamma' \in M(f^k)$ for some $k \in \mathbb{N}$. Suppose there is a $\Gamma \in M(f)$ in the same simplex S as Γ' such that Γ and Γ' have the same candidate NPEs.

Lemma 3.14 *Endow Γ and Γ' with the minimal train track structure w.r.t. f and f^k, respectively. Then if a turn is legal in Γ, it is legal in Γ'.*

Proof. Suppose t is a legal turn in Γ. Then $\exists \nu$, a legal candidate loop in Γ, such that $\exists M \in \mathbb{N}$ such that $f_S^m(\nu)$ crosses t $\forall m \geq M$. By Lemma 2.4, $\exists s \in \mathbb{N}$ such that $(f_S^k)^s(\nu)$ is legal in Γ' or contains an NP and such that $ks > M$. Since $f_S^{ks}(\nu)$ does not have an NP in Γ, and Γ and Γ' have the same candidate NPEs, $(f_S^{ks}(\nu))$ does not have an NP in Γ'. Take $m \in \mathbb{N}$. Then $(f_S^k)^m(f_S^{ks}(\nu)) = f_S^{km+ks}(\nu)$, so it crosses t. This means we have a legal loop $(f_S^{ks}(\nu))$ in Γ' the image of which under any power of f_S^k crosses t. Hence t is a legal turn in Γ'.

■

Lemma 3.15 *Let μ be a candidate loop in Γ'. Suppose $\sigma_{\Gamma',f}(\mu) = d$. Then $d \leq \lambda$.*

Proof. In Γ, $f_S(\mu)$ is legal other than NPs, because of the way we constructed f. So in Γ, $f_S(\mu)$ is the concatenation of legal segments and NPs $\{\beta_i\}_{i=1}^s$. Since Γ and Γ' have the same candidate NPEs, $f_S(\mu)$ in Γ' has NPs $\{\beta_i'\}_{i=1}^s$ corresponding to $\{\beta_i\}_{i=1}^s$. If t is a turn in $f_S(\mu)$ that is not the illegal turn of an NP in Γ', since Γ has the same candidate NPEs, it is not the illegal turn of an NP in Γ. This means t is a legal turn in Γ, and by Lemma 3.14, it is legal in Γ'. This means $f_S(\mu)$ is legal in Γ' other than the illegal turns taken by its NPs. Furthermore, since there are no PNPs, in Γ, μ has the same NPs as $f_S(\mu)$. This follows directly from the definition of PNPs, since the NPs in μ map to themselves under f^n, so $f_S(\mu)$ has at least all the same NPs, and since there are no PNPs, no part of the loop can map to a new NP. Again, due to the same candidate NPEs, in Γ', μ has the NPs $\{\beta_i'\}_{i=1}^s$. Note, however, that in μ, the segments between the NPs might not be legal in Γ or Γ'.

Let $c = \sum_{i=1}^{s} l_{\Gamma'}(\beta_i')$ be the combined length of all the NPs in Γ'. Then $\forall m \in \mathbb{N}$, we have $l_{\Gamma'}(f_S^{mk}(f_S(\mu)) = \lambda^{mk}(l_{\Gamma'}(f_S(\mu)) - c) + c = \lambda^{mk}(dl_{\Gamma'}(\mu) - c) + c$. This means $l_{\Gamma'}(f_S^{mk+1}(\mu)) = \lambda^{mk}(dl_{\Gamma'}(\mu) - c) + c$.

Let $T_m = f_S^{mk}(\mu)$. Since there are no PNPs, the set of NPs that T_m contains is $\{\beta_i\}_{i=1}^s$. From the way we constructed f, T_m is a concatenation of legal paths $\{L_{i,m}\}_{i=1}^t$ and the NPs $\{\beta_i\}_{i=1}^s$.

Claim 3.16 *In Γ, the stretch by f_Γ on $L_{i,m}$ approaches λ, i.e., $l_\Gamma(f_\Gamma(L_{i,m})) \to \lambda l_\Gamma(L_{i,m})$ as $m \to \infty$.*

Proof. Consider the lamination Λ corresponding to the train track map on Γ. Then each $L_{i,m}$ is a leaf segment, and $l_\Gamma(L_{i,m}) \to \infty$ as $m \to \infty$. Now by Lemma 2.7 in [BFH97], $l_\Gamma(f_\Gamma(L_{i,m})) \to \lambda l_\Gamma(L_{i,m})$ as $m \to \infty$.

∎

Since the only difference between Γ and Γ' is the metric, and in Γ and Γ' as marked graphs, the loops T_m and $f_S(T_m)$ are the same, they have the same NPEs: $\{\beta_i\}_{i=1}^s$ and $\{\beta_i'\}_{i=1}^s$ in Γ and Γ', respectively. Let $L'_{i,m}$ be the segment between the NPs in Γ' corresponding to $L_{i,m}$.

Claim 3.17 *In Γ', $l_{\Gamma'}(f_{\Gamma'}(L'_{i,m})) \to \lambda l_{\Gamma'}(L'_{i,m})$ as $m \to \infty$.*

Proof. As explained in the proof of Lemma 2.7 in [BFH97], in a sufficiently long leaf L of Λ, edges appear with a definite frequency. For an edge $e \in \Gamma$, call this frequency F_e, so that $N_e(L) \sim F_e l_\Gamma(L)$, where $N_e(L)$ is the number of times the edge e appears in L.

Now, for large enough m, $N_e(f_\Gamma(L_{i,m})) \sim l_\Gamma(f_\Gamma(L_{i,m}))F_e \sim \lambda l_\Gamma(L_{i,m})F_e \sim \lambda N_e(L_{i,m})$.

Since the lengths of the NPs don't change, while $l_\Gamma(L_{i,m})$ and $l_{\Gamma'}(L'_{i,m})$ approach infinity, $N_e(L'_{i,m}) \to N_e(L_{i,m})$ and $N_e(f_{\Gamma'}(L'_{i,m})) \to N_e(f_\Gamma(L_{i,m}))$ as $m \to \infty$. Then $N_e(f_{\Gamma'}(L'_{i,m})) \sim \lambda N_e(L'_{i,m})$

Now

$$l_{\Gamma'}(f_{\Gamma'}(L'_{i,m})) = \sum_{e \in E(\Gamma)} N_e(f_{\Gamma'}(L'_{i,m}))l_{\Gamma'}(e) \sim \sum_{e \in E(\Gamma)} \lambda N_e(f_{\Gamma'}(L'_{i,m}))l_{\Gamma'}(e) = \lambda l_{\Gamma'}(L'_{i,m})$$

∎

Since $\Gamma' \in M(f^k)$, the segments in μ between the NPs stretch by at most λ^{mk} when we apply f^{mk}, so $\sum_{i=1}^t l_{\Gamma'}(L'_{i,m}) \leq \lambda^{mk}(l_{\Gamma'}(\mu) - c)$.

Now $l_{\Gamma'}(f_S^{mk+1}(\mu)) \sim \lambda \sum_{i=1}^t l_{\Gamma'}(L'_{i,m}) + c \leq \lambda(\lambda^{mk}(l_{\Gamma'}(\mu) - c)) + c = \lambda^{mk}(\lambda l_{\Gamma'}(\mu) - \lambda c) + c$

However, from the beginning of the proof, we have $l_{\Gamma'}(f_S^{mk+1}(\mu)) = \lambda^{mk}(d l_{\Gamma'}(\mu) - c) + c$.

This means $\lambda l_{\Gamma'}(\mu) - \lambda c = d l_{\Gamma'}(\mu) - c$, so

$$d = \frac{\lambda l_{\Gamma'}(\mu) - \lambda c + c}{l_{\Gamma'}(\mu)} \leq \lambda$$

with equality if and only if $c = 0$, i.e., if μ does not contain any NPs.

Figure 3.14 shows the commutative diagram that is the essence of this proof.

■

The Lemmas above give the following Proposition:

Proposition 3.18 *Suppose $\Gamma' \in M(f^k)$ for some $k \in \mathbb{N}$. Suppose there is a $\Gamma \in M(f)$ in the same simplex S as Γ' such that Γ and Γ' have the same candidate NPEs. Then $\Gamma' \in M(f)$.*

Proof. From Lemma 3.15, we know that the stretch on any candidate loop is less than or equal to λ.

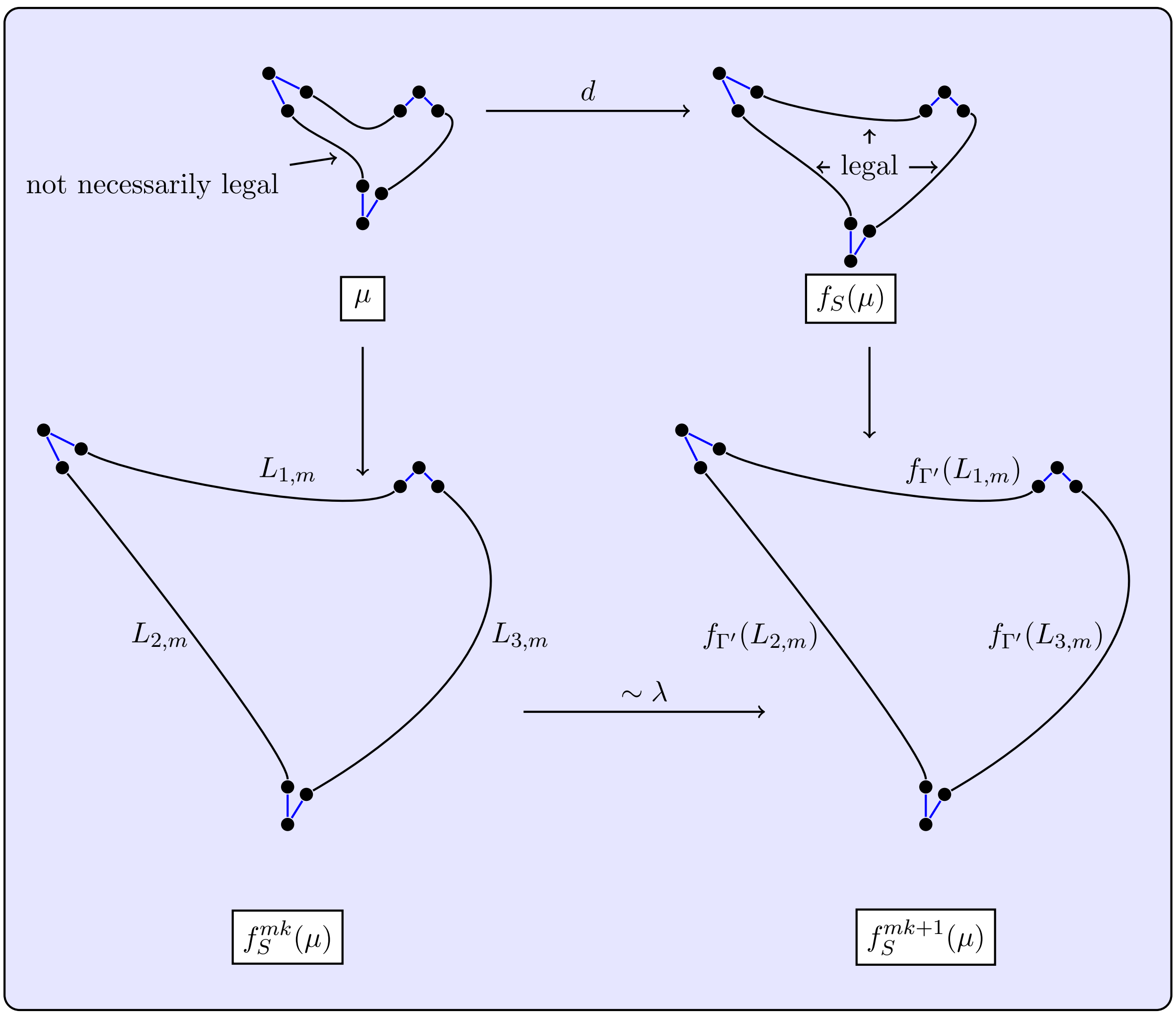

Figure 3.14. Commutative diagram for no PNP

By the definition of displacement as the maximal stretch on any candidate loop, it follows that $\Gamma' \in M(f)$.

∎

We can now prove the Theorem:

Proof.[proof of Theorem 3.3] Take a simplex S such that $\exists k$ with $M(g^k) \cap S \neq \varnothing$ and g^k is a special power for f. Let C be a combination of candidate NPEs such that there exists $\Gamma \in S$ with that combination such that $\Gamma \in M(g^k)$ for some k. Let $k_{S,C}$ be the smallest power of g such that $g^{k_{S,C}}$ is a special power and $M(g^{k_{S,C}}) \cap S$ contains a graph with the candidate NPE combination C. Then, by proposition 3.18, for any $m \in \mathbb{N}$, $M(g^{mk_{S,C}}) = M(g^{k_{S,C}})$. By Lemma 3.13, there are finitely many candidate NPEs, so there are finitely many possible combinations of candidate NPEs for graphs in S. We can take the composition of $g^{k_{S,C}}$ to get a g^{k_S} such that $M(g^{mk_S}) = M(g^{k_S})$. Since there are finitely many g-orbits of simplices that intersect the minset of any power of g, we can take the product of their k_S's to get a power $g^{\prod_S k_S}$ such that $\forall m \in \mathbb{N}$, $M((g^{\prod_S k_S})^m) = M(g^{\prod_S k_S})$.

∎

CHAPTER 4

NIELSEN AND PRE-NIELSEN PATHS IN HYPERBOLIC IRREDUCIBLE AUTOMORPHISMS

This chapter shows what ranks and indices of hyperbolic irreducible automorphisms have NPs and PNPs, hence showing which automorphisms have powers with minsets that do or do not stabilize under further powers. Most of the proofs are by example, showing that all three cases are possible for almost every rank and index. We summarize these results in Figure 4.1. The specific examples are references in Figure 4.2, Figure 4.3, and Figure 4.4.

First, we give a proof that for any type of automorphism, the stable part of the minset does not have PNPs, so it has an eventually stable minset.

Corollary 4.1 *Let g be a hyperbolic irreducible automorphism. Let $SM(g)$ be the set of stable representatives in $M(g)$. Then there is an $m \in \mathbb{N}$ such that $\forall k$, $SM(g^{mk}) = SM(g^m)$.*

Proof. Let Γ be in $SM(g)$. Now there are two cases. If g is ageometric, then Γ does not have an NP. So no representative in $SM(g)$ has an NP, meaning $SM(g^k)$ eventually stabilizes. If g is parageometric or geometric, then Γ has exactly one NP, and by [BFH97], the illegal turn of the NP is the only illegal turn in Γ. Since a PNP would give a second illegal turn, Γ does not contain a PNP. Hence, no representative in $SM(g)$ has a PNP, meaning $SM(g^k)$ eventually stabilizes.

∎

Note also that the stability of minset proofs were contained in single simplices, so we can state the results in a particular simplex regardless of an automorphism's behavior elsewhere.

Corollary 4.2 *Let g be a hyperbolic irreducible automorphism. If S is a simplex in $\mathcal{X}_n$ such that no train track representative of g in S contains a PNP, then there is an automorphism f, a power of g, such that $\forall k$, $M(f^k) \cap S = M(f) \cap S$.*

i \ n	2	3	≥ 4
0		S	S
$-\dfrac{1}{2}$		S U	S U
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2-n$		S U	S U
$\dfrac{3}{2}-n$		S U	S U
$1-n$	S U	S U	S U

Figure 4.1. Stable and unstable minset in hyperbolic irreducible automorphisms

Here is a summary of what we show for hyperbolic irreducible automorphisms:

- Proof that if $i = 0$, g cannot have an NP (Lemma 4.3)

- A list of examples with a PNP, hence a minset that never stabilizes for $n \geq 2$ and $-\frac{1}{2} \geq i \geq 1 - n$ (excluding $n = 2$ with $i = -.5$, since rank 2 automorphisms are geometric).

- A list of examples that have an NP but appear to not have a PNP for $n \geq 2$ and $-\frac{1}{2} \geq i \geq 1 - n$ (excluding $n = 2$ with $i = -.5$)

- A list of examples that never have an NP, hence an eventually stable minset, for $n \geq 3$ and $0 \geq i \geq 1.5 - n$. By Lemma 3.1 of [HM08], an NP gives a cut vertex in $IW(g)$. All these examples have an Ideal Whitehead Graph with no cut vertices, hence no NP. We are excluding $n = 2$ and the index $i = 1 - n$, since every representative for a geometric or parageometric automorphism has an NP.

i \ n	2	3	4	5	6	7	≥ 8
$-.5$		6.22	6.18	6.19	6.19	6.19	6.19
-1	6.25	6.23	6.16	6.17	6.17	6.17	6.17
-1.5		6.24	6.1	6.2	6.2	6.2	6.2
-2			6.3	6.4	6.4	6.4	6.4
-2.5			6.20	6.10	6.9	6.9	6.9
-3			6.26	6.5	6.8	6.8	6.8
-3.5				6.21	6.11	6.14	6.14
-4				6.28	6.15	6.7	6.7
-4.5					6.21	6.12	6.13
$\frac{2-k}{6 \leq k \leq n}$						6.7	6.7
$\frac{2.5-k}{7 \leq k \leq n}$							6.13
$2.5-n$		6.22	6.1	6.10	6.11	6.12	6.12
$2-n$		6.23	6.3	6.5	6.6	6.6	6.6
$1.5-n$		6.24	6.20	6.21	6.21	6.21	6.21
$1-n$ para			6.26	6.28	6.28	6.28	6.28
$1-n$ geo	6.25		6.27	6.28	6.28	6.28	6.28

Figure 4.2. Examples with a PNP

Lemma 4.3 *Suppose g is a hyperbolic automorphism with $i(g) = 0$. Then g cannot have an NP, so it has a power with a stable minset.*

Proof. By Lemma 3.1 of [HM08], an NP gives a cut vertex in $IW(g)$. When $i(g) = 0$, $IW(g)$ is empty (or consists of components with two vertices, depending on how you define it), hence no cut vertex and no NP.

∎

i \ n	2	3	4	5	6	7	≥ 8
$-.5$			6.30	6.35	6.35	6.35	6.35
-1	6.49	6.48	6.38	6.39	6.39	6.39	6.39
-1.5		6.47	6.40	6.41	6.41	6.41	6.41
-2		6.51	6.29	6.31	6.31	6.31	6.31
-2.5			6.45	6.42	6.34	6.34	6.34
-3			6.50	6.32	6.33	6.33	6.33
-3.5				6.46	6.43	6.44	6.44
$\frac{2.5-k}{7 \leq k < n}$						6.44	6.44
$\frac{2-k}{6 \leq k < n}$						6.37	6.37
$\frac{2.5-k}{7 \leq k < n}$						6.44	6.44
$2.5-n$			6.40	6.42	6.43	6.43	6.43
$2-n$		6.48	6.29	6.32	6.36	6.36	6.36
$1.5-n$		6.47	6.45	6.46	6.46	6.46	6.46
$\frac{1-n}{para}$		6.28	6.28	6.28	6.28	6.28	6.28
$\frac{1-n}{geo}$	6.49	6.51	6.50	6.28	6.28	6.28	6.28

Figure 4.3. Examples with a NP but seemingly no PNP

i \ n	3	4	5	6	7	≥ 8
0	6.74	6.72	6.73	6.73	6.73	6.73
$-.5$	6.76	6.71	6.75	6.75	6.75	6.75
-1	6.78	6.52	6.53	6.54	6.54	6.54
-1.5	6.77	6.61	6.62	6.62	6.62	6.62
-2		6.55	6.56	6.57	6.57	6.57
-2.5		6.63	6.64	6.64	6.64	6.64
-3			6.69	6.70	6.70	6.70
-3.5			6.67	6.68	6.68	6.68
$\frac{2-k}{6 \leq k < n}$					6.60	6.60
$\frac{1.5-k}{6 \leq k < n}$					6.66	6.66
$2-n$	6.78	6.55	6.69	6.59	6.59	6.59
$1.5-n$	6.77	6.63	6.67	6.65	6.65	6.65

Figure 4.4. Examples with no NP

CHAPTER 5

OTHER RESULTS

This chapter contains some results that are not part of the main focus of the paper.

5.1 Minset is Contractible in a Simplex

Theorem 5.1 *Let f be a hyperbolic iwip automorphism. Let S be a simplex in $\mathcal{X}_n$. Then if $M(f) \cap S \neq \varnothing$, $M(f) \cap S$ is contractible.*

To prove this, we will define a cone field on S by taking at each point in S the cone of directions of nonincreasing displacement. We can then show that the cone field function is lower semicontinuous, so by the Michael Selection Theorem, we can pick a continuous vector field on S, where each vector gives a nonincreasing direction. This means that if $M(f) \cap S \neq \varnothing$, the vector field gives a deformation retract from from S to $M(f) \cap S$, showing $M(f) \cap S$ is contractible.

First, we do the preliminary work to help define the cone field. Let λ be the stretch constant of f. Since S is a manifold with boundary, we can define the tangent bundle $TS = \bigsqcup_{\Gamma \in S} T_\Gamma S$.

Lemma 5.1 *Let $\Gamma_0 \in S$. Then there is a nonempty cone C_{Γ_0} of directions in $T_{\Gamma_0}S$ s.t. if $d \in C_{\Gamma_0}$, then the displacement function $\tilde{f}$ is nonincreasing along d.*

Proof. Let Γ be a point in S. For each candidate loop μ in Γ, we can find the stretch of μ under f, $\sigma_{\Gamma,f}(\mu) = \dfrac{l_\Gamma(f_S(\mu))}{l_\Gamma(\mu)}$. Then the displacement function $\tilde{f}$ at Γ is equal to $\max_{\mu \text{ candidate}} \sigma_{\Gamma,f}(\mu)$. Let $\{e_i\}_{i=1}^s$ be the set of edges in Γ. Let μ_j be a candidate loop in Γ. Let $c_{j,i}$ be the number of times μ_j crosses e_i and $d_{j,i}$ be the number of times $f_S(\mu_j)$ crosses e_i. Then $\sigma_{\Gamma,f}(\mu_j) = \dfrac{\sum_{i=1}^s d_{j,i} l_\Gamma(e_i)}{\sum_{i=1}^s c_{j,i} l_\Gamma(e_i)}$. Now suppose $\{\mu_j\}_{j \in \mathcal{A}}$ is the set of candidate loops in Γ. Then $\tilde{f}(\Gamma) = \max_{\mu_j \in \mathcal{A}} \dfrac{\sum_{i=1}^s d_{j,i} l_\Gamma(e_i)}{\sum_{i=1}^s c_{j,i} l_\Gamma(e_i)}$. Note that if we ignore the metric, a different graph Γ' in S has all the same loops μ_j and edges e_i (some of the e_i might have length zero). Then

$$\tilde{f}(\Gamma') = \max_{\mu_j \in \mathcal{A}} \frac{\sum_{i=1}^{s} d_{j,i} l'_\Gamma(e_i)}{\sum_{i=1}^{s} c_{j,i} l'_\Gamma(e_i)}$$

Now fix a specific Γ_0 in S. For a maximally stretched candidate loop μ_j in Γ_0,

$$\sigma_{\Gamma,f}(\mu_j) = \frac{\sum_{i=1}^{s} d_{j,i} l_\Gamma(e_i)}{\sum_{i=1}^{s} c_{j,i} l_\Gamma(e_i)}$$

is a fractional linear function, so in $T_{\Gamma_0}S$, there is a codimension one halfspace H_{Γ_0,μ_j} of directions along which $\sigma_{\Gamma,f}(\mu_j)$ is nonincreasing. Let $\mathcal{A}_{\Gamma_0}$ be the set of maximally stretched candidate loops in Γ_0. Let S_j be the set of graphs in which μ_j is a maximally stretched loop. This means $\Gamma_0 \in \bigcap_{\mu_j \in \mathcal{A}_{\Gamma_0}} S_j$. Since each S_j is closed by Lemma 5.2, there is an open neighborhood of Γ_0, call it U_{Γ_0}, s.t. if $S_j \cap U_{\Gamma_0} \neq \varnothing$, then $\Gamma_0 \in S_j$. This means, if $\Gamma \in U_{\Gamma_0}$, $\mathcal{A}_\Gamma \subseteq \mathcal{A}_{\Gamma_0}$, i.e., no graph in the neighborhood has any new maximally stretched loops. Now $\bigcap_{\mu_j \in \mathcal{A}_{\Gamma_0}} H_{\Gamma_0,\mu_j}$ gives a cone C_{Γ_0} in $T_{\Gamma_0}S$ of directions along which each $\sigma_{\Gamma,f}(\mu_j)$ is nonincreasing. Since there is a neighborhood with no new maximally stretched loops, $\tilde{f}$ is nonincreasing along each direction in C_{Γ_0}. By the proof of Proposition 6 in [Bes11], there is a point arbitrarily close to Γ_0 on which the value of the displacement function is smaller than or equal to that on Γ_0. This means C_{Γ_0} is nonempty.

$\blacksquare$

Now, we can define the cone field by

$$\begin{array}{rccc} \mathcal{F}: & S & \longrightarrow & TS \\ & \Gamma_0 & \longmapsto & C_{\Gamma_0} \end{array}$$

Lemma 5.2 *Consider a candidate loop μ_j in the marked graph representing S. Then the set S_j of graphs in which μ_j is a maximally stretched loop is closed.*

Proof. Take another candidate loop μ_k. Then $\sigma_{\Gamma,f}(\mu_j)$ and $\sigma_{\Gamma,f}(\mu_k)$ are continuous functions on S (since in the proof of Lemma 5.1 we show that they are fractional linear functions), so the set $S_{j,k}$ where $\sigma_{\Gamma,f}(\mu_j) \geq \sigma_{\Gamma,f}(\mu_k)$ is closed. The loop μ_j is maximally stretched in a graph Γ whenever $\sigma_{\Gamma,f}(\mu_j) \geq \sigma_{\Gamma,f}(\mu_k)$ for any other candidate loop μ_k. This means $S_j = \bigcap_{\mu_k,\ \text{candidate loop}} S_{j,k}$. Since this is an intersection of closed sets, S_j is closed.

$\blacksquare$

Lemma 5.3 *The multivalued function $\mathcal{F}$ is lower semicontinuous.*

Proof. First, the formal definition of lower semicontinuous. A multivalued function $Y : A \longrightarrow B$ is said to be lower semicontinuous at a point $a \in A$ if for any open set V intersecting

$Y(A)$, there exists a neighborhood U of a s.t. for all $u \in U$, $Y(U)$ intersects V. This means that for a function that is continuous other than places where the image set suddenly grows or decreases, each point has an open neighborhood where the image set might increase but will not decrease.

Let $\Gamma_0 \in S$. Recall that $\mathcal{A}_{\Gamma_0}$ is the set of loops maximally stretched in Γ_0. This means $\Gamma_0 \in \bigcap_{\mu_j \in \mathcal{A}_{\Gamma_0}} S_j$. Since each S_j is closed, there is an open neighborhood of Γ_0, call it U_{Γ_0}, s.t. if $S_j \cap U_{\Gamma_0} \neq \varnothing$, then $\Gamma_0 \in S_j$. This means, if $\Gamma \in U_{\Gamma_0}$, $\mathcal{A}_\Gamma \subseteq \mathcal{A}_{\Gamma_0}$, i.e., no graph in the neighborhood has any new maximally stretched loops (this is the same neighborhood we used in the proof of Lemma 5.1). This implies that $\bigcap_{\mu_j \in \mathcal{A}_\Gamma} H_{\Gamma,\mu_j} \supseteq \bigcap_{\mu_j \in \mathcal{A}_{\Gamma_0}} H_{\Gamma,\mu_j}$, meaning that in U_{Γ_0} the image cone under $\mathcal{F}$ might increase but will not decrease. This is precisely what we need for the lower semicontinuity of $\mathcal{F}$.

■

Now we can prove the main theorem of this section.

Proof.[proof of Theorem 5.1]

According to the Michael Selection Theorem, if we have a lower semicontinuous multivalued function from a Banach space to a paracompact space s.t. the image values are nonempty, convex, and closed, then there is continuous selection of a single-valued function.

Consider the cone field $\mathcal{F} : S \longrightarrow TS$ defined above. Since S and TS are manifolds, S is a Banach space and TS is paracompact. The image of each point under $\mathcal{F}$ is a nonempty cone, hence it is nonempty, convex, and closed. This means that by the Michael Selection Theorem, we can choose a continuous function $S \longrightarrow TS$. Since each cone consists of directions along which the displacement function $\tilde{f}$ is nonincreasing, we get a vector field on S of directions along which $\tilde{f}$ is nonincreasing. This means that if $M(f) \cap S \neq \varnothing$, the vector field gives a deformation retract from from S to $M(f) \cap S$, showing $M(f) \cap S$ is contractible.

■

5.2 Miscellaneous

Handel and Mosher's paper *Axes in Outer Space* [HM08] was a very useful resource in writing this paper. In that work, there are several questions that are connected to the results in this paper.

Question 5 of Section 1.5 in [HM08] asks what values of $i(f) - (1 - n)$ are possible, where f is a hyperbolic iwip automorphism in $Out(F_n)$ and $i(f)$ is the index of f. It also asks whether $\max\{i(f)-(1-n)\} \longrightarrow \infty$ as $n \longrightarrow \infty$. The examples in this paper show the following:

Lemma 5.4 *Let f be a hyperbolic iwip automorphism in $Out(F_n)$ and let $i(f)$ be the index of f. The possible values for $i(f)-(1-n)$ are $0 \leq (i(\varphi)-(1-n)) \leq n-1$, so all the potential values are achieved and* $\max\{i(f)-(1-n)\} \longrightarrow \infty$ *as* $n \longrightarrow \infty$.

In Section 1.1 of [HM08], it is conjectured that $\exists N$ s.t. the axis bundle of a hyperbolic iwip automorphism f is the closure of $\bigcup_{i=1}^{N} TT(f^i)$, where $TT(f^i)$ is the set of train track representatives of f. From the results in this paper, it appears that if f has a PNP, hence a minset that increases with powers and never stabilizes, then the union would actually need to be infinite.

The examples also give a partial answer to Question 7 in [HM08], which asks what types of ideal Whitehead graphs are possible for hyperbolic iwip automorphisms.

CHAPTER 6

SPECIFIC EXAMPLES

Unless otherwise stated, the map given is on a rose with edges marked by the generators $a, b, c, \ldots$.

6.1 PNP

Example 6.1 $n = 4$, $i = -1.5$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aad \\ b & \longrightarrow & bd \\ c & \longrightarrow & a \\ d & \longrightarrow & cb \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bd \\ c & \longrightarrow & ab \\ d & \longrightarrow & cb \end{array}$$

Ideal Whitehead Graph: Figure 6.1

Example 6.2 $n \geq 5$, $i = -1.5$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aad \\ b & \longrightarrow & bd \\ c & \longrightarrow & a \\ d & \longrightarrow & ecb \\ e & \longrightarrow & fbc \\ & \vdots & \\ a_{n-1} & \longrightarrow & a_n b a_{n-2} \\ a_n & \longrightarrow & b a_{n-1} \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bd \\ c & \longrightarrow & ab \\ d & \longrightarrow & ecb \\ e & \longrightarrow & fbc \\ & \vdots & \\ a_{n-1} & \longrightarrow & a_n b a_{n-2} \\ a_n & \longrightarrow & b a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.2

Example 6.3 $n = 4$, $i = -2$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aad \\ b & \longrightarrow & bd \\ c & \longrightarrow & a \\ d & \longrightarrow & dcb \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bd \\ c & \longrightarrow & ab \\ d & \longrightarrow & dcb \end{array}$$

Ideal Whitehead Graph:Figure 6.3

Example 6.4 $n \geq 5$, $i = -2$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aad \\ b & \longrightarrow & bd \\ c & \longrightarrow & be \\ d & \longrightarrow & a_n ad \\ e & \longrightarrow & f \\ & \vdots & \\ a_{n-1} & \longrightarrow & a_n \\ a_n & \longrightarrow & c \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{d}\dot{B}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bd \\ c & \longrightarrow & be \\ d & \longrightarrow & a_n abd \\ e & \longrightarrow & f \\ & \vdots & \\ a_{n-1} & \longrightarrow & a_n \\ a_n & \longrightarrow & c \end{array}$$

Ideal Whitehead Graph: Figure 6.4

Example 6.5 $n = 5$, $i = -3$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aad \\ b & \longrightarrow & bd \\ c & \longrightarrow & a \\ d & \longrightarrow & ecb \\ e & \longrightarrow & de \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bd \\ c & \longrightarrow & ab \\ d & \longrightarrow & ecb \\ e & \longrightarrow & de \end{array}$$

Ideal Whitehead Graph: Figure 6.5

Example 6.6 $n \geq 5$, $i = 2 - n$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aad \\ b & \longrightarrow & bd \\ c & \longrightarrow & a \\ d & \longrightarrow & a_n b \\ e & \longrightarrow & dcbe \\ f & \longrightarrow & ebf \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1}ba_n \end{array}$$

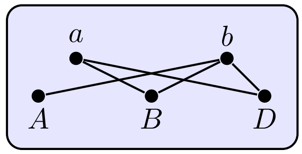

Figure 6.1. PNP: $n = 4$, $i = -1.5$

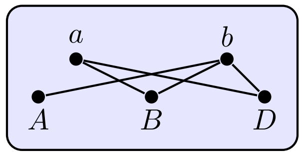

Figure 6.2. PNP: $n \geq 5$, $i = -1.5$

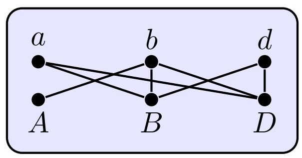

Figure 6.3. PNP: $n = 4$, $i = -2$

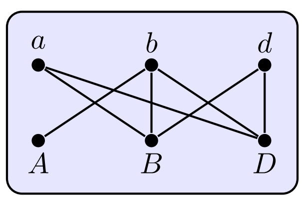

Figure 6.4. PNP: $n \geq 5$, $i = -2$

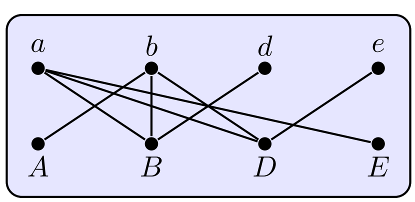

Figure 6.5. PNP: $n = 5$, $i = -3$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bd \\ c & \longrightarrow & ab \\ d & \longrightarrow & a_n b \\ e & \longrightarrow & dcbe \\ f & \longrightarrow & ebf \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1}ba_n \end{array}$$

Ideal Whitehead Graph: Figure 6.6

Example 6.7 $n \geq 7$, $6 \leq k < n$, $i = 2 - k$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aad \\ b & \longrightarrow & bd \\ c & \longrightarrow & a \\ d & \longrightarrow & a_k a_n b \\ e & \longrightarrow & dcbe \\ f & \longrightarrow & ebf \\ & \vdots & \\ a_k & \longrightarrow & a_{k-1}ba_k \\ a_{k+1} & \longrightarrow & c \\ a_{k+2} & \longrightarrow & a_{k+1} \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bd \\ c & \longrightarrow & ab \\ d & \longrightarrow & a_k a_n b \\ e & \longrightarrow & dcbe \\ f & \longrightarrow & ebf \\ & \vdots & \\ a_k & \longrightarrow & a_{k-1}ba_k \\ a_{k+1} & \longrightarrow c & \end{array}$$

$$\begin{array}{ccc} a_{k+2} & \longrightarrow & a_{k+1} \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.7

Example 6.8 $n \geq 6$, $i = -3$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aad \\ b & \longrightarrow & bd \\ c & \longrightarrow & a \\ d & \longrightarrow & ea_nb \\ e & \longrightarrow & dcbe \\ f & \longrightarrow & c \\ g & \longrightarrow & f \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bd \\ c & \longrightarrow & ab \\ d & \longrightarrow & ea_nb \\ e & \longrightarrow & dcbe \\ f & \longrightarrow & c \\ g & \longrightarrow & f \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.8

Example 6.9 $n \geq 6$, $i = -2.5$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aad \\ b & \longrightarrow & bd \\ c & \longrightarrow & a \\ d & \longrightarrow & ea_ncb \\ e & \longrightarrow & dc \\ f & \longrightarrow & e \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

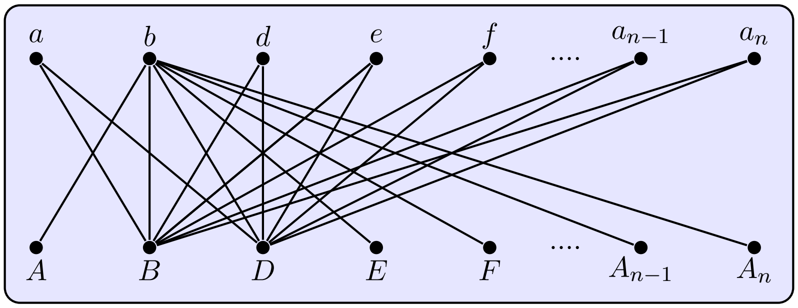

Figure 6.6. PNP: $n \geq 5$, $i = 2 - n$

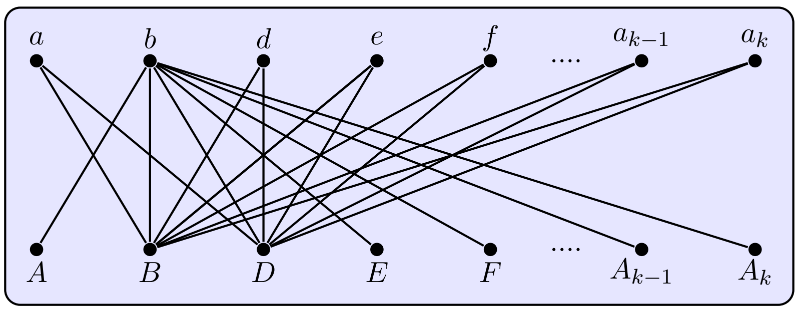

Figure 6.7. PNP: $n \geq 7$, $6 \leq k < n$, $i = 2 - k$

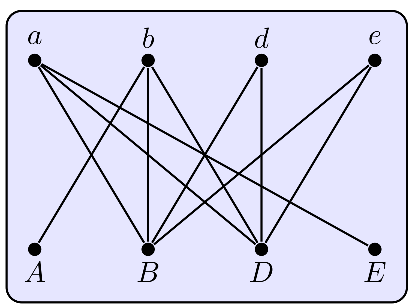

Figure 6.8. PNP: $n \geq 6$, $i = -3$

NP: $\dot{a}B$, *PNP:* $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bd \\ c & \longrightarrow & ab \\ d & \longrightarrow & ea_ncb \\ e & \longrightarrow & dc \\ f & \longrightarrow & e \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.9, Figure 6.10

Example 6.10 $n = 5$, $i = -2.5$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aad \\ b & \longrightarrow & bd \\ c & \longrightarrow & a \\ d & \longrightarrow & ecb \\ e & \longrightarrow & dc \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bd \\ c & \longrightarrow & ab \\ d & \longrightarrow & ecb \\ e & \longrightarrow & dc \end{array}$$

Ideal Whitehead Graph: Figure 6.11

Example 6.11 $n = 6$, $i = -3.5$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & ead \\ b & \longrightarrow & bd \\ c & \longrightarrow & a \\ d & \longrightarrow & dcfb \\ e & \longrightarrow & af \\ f & \longrightarrow & e \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ea \\ b & \longrightarrow & bd \\ c & \longrightarrow & ab \\ d & \longrightarrow & dcfb \\ e & \longrightarrow & abf \\ f & \longrightarrow & e \end{array}$$

Ideal Whitehead Graph: Figure 6.12

Example 6.12 $n \geq 7$, $i = 2.5 - n$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aad \\ b & \longrightarrow & bd \\ c & \longrightarrow & a \\ d & \longrightarrow & a_n cfb \\ e & \longrightarrow & dbe \\ f & \longrightarrow & ebf \\ g & \longrightarrow & fbg \\ & \vdots & \\ a_{n-1} & \longrightarrow & a_{n-2}ba_{n-1} \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bd \\ c & \longrightarrow & ab \\ d & \longrightarrow & a_n cfb \\ e & \longrightarrow & dbe \\ f & \longrightarrow & ebf \\ g & \longrightarrow & fbg \\ & \vdots & \\ a_{n-1} & \longrightarrow & a_{n-2}ba_{n-1} \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.13

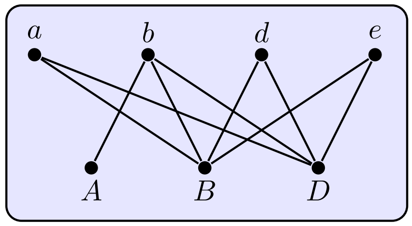

Figure 6.9. PNP: $n \geq 6$, $i = -2.5$, n odd

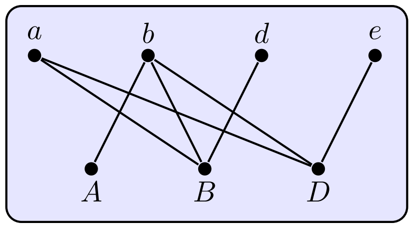

Figure 6.10. PNP: $n \geq 6$, $i = -2.5$, n even

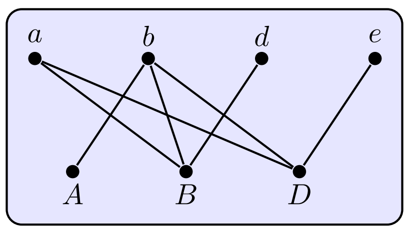

Figure 6.11. PNP: $n = 5$, $i = -2.5$

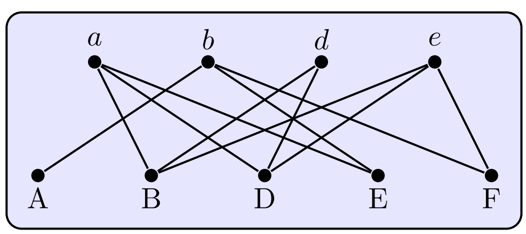

Figure 6.12. PNP: $n = 6$, $i = -3.5$

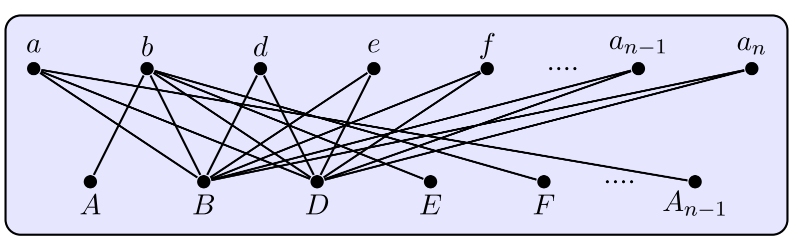

Figure 6.13. PNP: $n \geq 7$, $i = 2.5 - n$

Example 6.13 $n \geq 8,\ 7 \leq k < n,\ i = 2.5 - k$

Representative with PNP:

$$\begin{array}{ccc}
a & \longrightarrow & aad \\
b & \longrightarrow & bd \\
c & \longrightarrow & a \\
d & \longrightarrow & a_k cb \\
e & \longrightarrow & dbe \\
 & \vdots & \\
a_{k-1} & \longrightarrow & a_{k-2}ba_{k-1} \\
a_k & \longrightarrow & a_{k-1} \\
a_{k+1} & \longrightarrow & a_{k+2}a_k \\
 & \vdots & \\
a_{n-1} & \longrightarrow & a_n a_{n-2} \\
a_n & \longrightarrow & aa_{n-1}c
\end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc}
a & \longrightarrow & aba \\
b & \longrightarrow & bd \\
c & \longrightarrow & a \\
d & \longrightarrow & a_k cb \\
e & \longrightarrow & dbe \\
 & \vdots & \\
a_{k-1} & \longrightarrow & a_{k-2}ba_{k-1} \\
a_k & \longrightarrow & a_{k-1} \\
 & \vdots & \\
a_{n-1} & \longrightarrow & a_n a_{n-2} \\
a_n & \longrightarrow & aba_{n-1}c
\end{array}$$

Ideal Whitehead Graph: Figure 6.14

Example 6.14 $n \geq 7,\ i = -3.5$

Representative with PNP:

$$\begin{array}{ccc}
a & \longrightarrow & ead \\
b & \longrightarrow & bd \\
c & \longrightarrow & a \\
d & \longrightarrow & dcgfb \\
e & \longrightarrow & af \\
f & \longrightarrow & e \\
g & \longrightarrow & h \\
 & \vdots &
\end{array}$$

$$\begin{array}{ccc} a_{n-1} & \longrightarrow & a_n \\ a_n & \longrightarrow & c \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ea \\ b & \longrightarrow & bd \\ c & \longrightarrow & ab \\ d & \longrightarrow & dcgfb \\ e & \longrightarrow & abf \\ f & \longrightarrow & e \\ g & \longrightarrow & h \\ & \vdots & \\ a_{n-1} & \longrightarrow & a_n \\ a_n & \longrightarrow & c \end{array}$$

Ideal Whitehead Graph: Figure 6.15

Example 6.15 $n = 6$, $i = -4$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aad \\ b & \longrightarrow & bd \\ c & \longrightarrow & a \\ d & \longrightarrow & fab \\ e & \longrightarrow & dcf \\ f & \longrightarrow & e \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{D}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bd \\ c & \longrightarrow & ab \\ d & \longrightarrow & fabb \\ e & \longrightarrow & dcf \\ f & \longrightarrow & e \end{array}$$

Ideal Whitehead Graph: Figure 6.16

Example 6.16 $n = 4$, $i = -1$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aac \\ b & \longrightarrow & bc \\ c & \longrightarrow & adbc \\ d & \longrightarrow & a \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{a}\dot{C}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bc \\ c & \longrightarrow & abdbc \\ d & \longrightarrow & ab \end{array}$$

Ideal Whitehead Graph: Figure 6.17

Example 6.17 $n \geq 5$, $i = -1$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aac \\ b & \longrightarrow & bc \\ c & \longrightarrow & aa_nbc \\ d & \longrightarrow & a \\ e & \longrightarrow & d \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{a}\dot{C}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bc \\ c & \longrightarrow & aba_nbc \\ d & \longrightarrow & ab \\ e & \longrightarrow & d \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.18

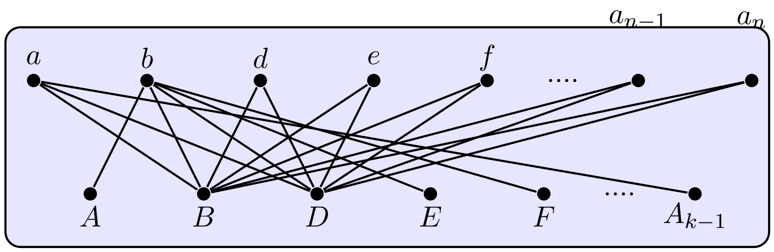

Figure 6.14. PNP: $n \geq 8$, $7 \leq k < n$, $i = 2.5 - k$

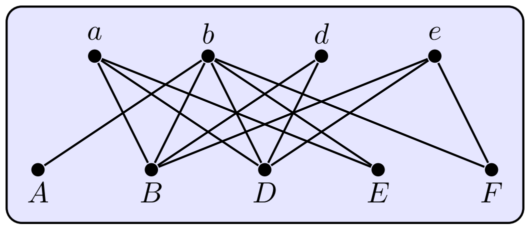

Figure 6.15. PNP: $n \geq 7$, $i = -3.5$

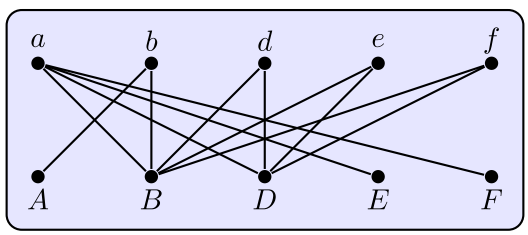

Figure 6.16. PNP: $n = 6$, $i = -4$

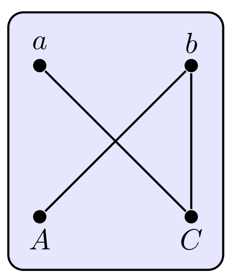

Figure 6.17. PNP: $n = 4$, $i = -1$

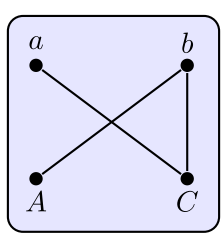

Figure 6.18. PNP: $n \geq 5$, $i = -1$

Example 6.18 $n = 4,\ i = -.5$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & dac \\ b & \longrightarrow & bc \\ c & \longrightarrow & ac \\ d & \longrightarrow & b \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{B}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & da \\ b & \longrightarrow & bc \\ c & \longrightarrow & abc \\ d & \longrightarrow & b \end{array}$$

Ideal Whitehead Graph: Figure 6.19

Example 6.19 $n \geq 5,\ i = -.5$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & dac \\ b & \longrightarrow & bc \\ c & \longrightarrow & a_n ac \\ d & \longrightarrow & b \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{B}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & da \\ b & \longrightarrow & bc \\ c & \longrightarrow & a_n abc \\ d & \longrightarrow & b \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.20

Example 6.20 $n = 4,\ i = -2.5$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & dad \\ b & \longrightarrow & cbd \end{array}$$

$$\begin{array}{ccc} c & \longrightarrow & ad \\ d & \longrightarrow & b \end{array}$$

NP: $\dot{a}\dot{B}$

PNP: $\dot{c}\dot{B}$

Stable Representative:

Same map as above, but on the graph with the edges labeled a *and* b *partially folded (Figure 6.21).*

Ideal Whitehead Graph: Figure 6.22

Example 6.21 $n \geq 5$, $i = 1.5 - n$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & a_n ad \\ b & \longrightarrow & cbd \\ c & \longrightarrow & ad \\ d & \longrightarrow & b \\ e & \longrightarrow & de \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1}a_n \end{array}$$

NP: $\dot{a}\dot{B}$

PNP: $\dot{c}\dot{B}$

Stable Representative:

Same map as above, but on the graph with the edges labeled a *and* b *partially folded (Figure 6.23).*

Ideal Whitehead Graph: Figure 6.24

Example 6.22 $n = 3$, $i = -.5$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & bac \\ b & \longrightarrow & bc \\ c & \longrightarrow & ac \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{B}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ba \\ b & \longrightarrow & bc \\ c & \longrightarrow & abc \end{array}$$

Ideal Whitehead Graph: Figure 6.25

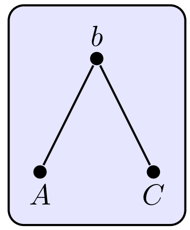

Figure 6.19. PNP: $n = 4$, $i = -.5$

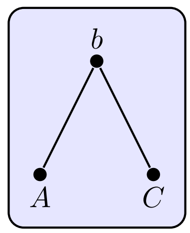

Figure 6.20. PNP: $n \geq 5$, $i = -.5$

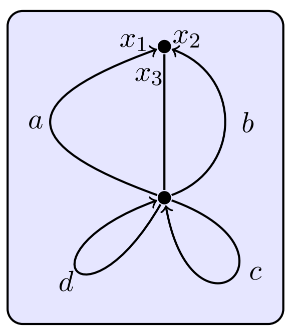

Figure 6.21. Graph on 4 generators with $\dot{a}\dot{B}$ folded

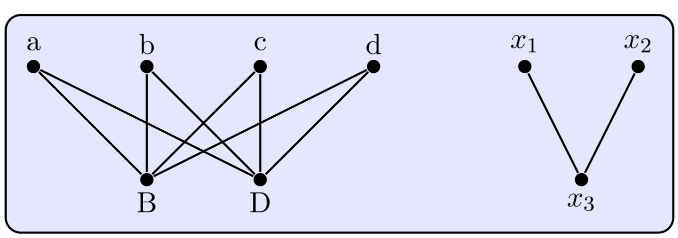

Figure 6.22. PNP: $n = 4$, $i = -2.5$

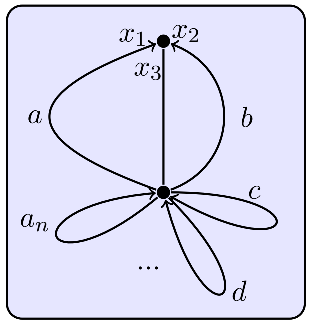

Figure 6.23. Graph on 4 generators with $\dot{a}\dot{B}$ folded

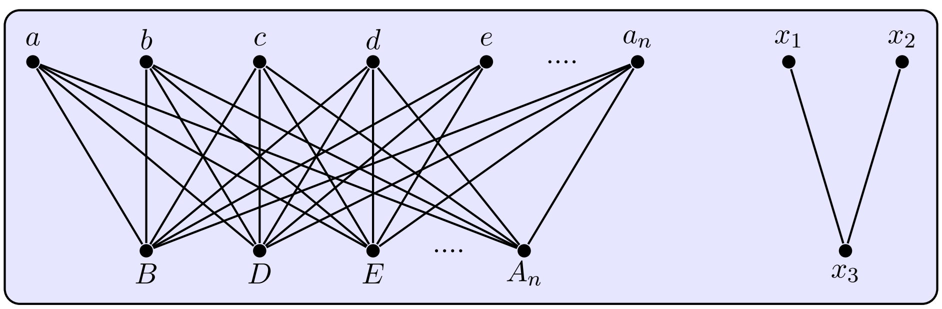

Figure 6.24. PNP: $n \geq 5$, $i = 1.5 - n$

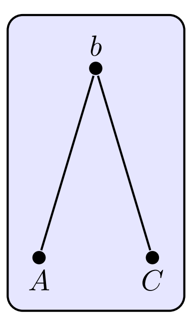

Figure 6.25. PNP: $n = 3$, $i = -.5$

Example 6.23 $n = 3,\ i = -1$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & aac \\ b & \longrightarrow & bc \\ c & \longrightarrow & abc \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{a}\dot{C}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bc \\ c & \longrightarrow & abbc \end{array}$$

Ideal Whitehead Graph:Figure 6.26

Example 6.24 $n = 3,\ i = -1.5$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & cbcac \\ b & \longrightarrow & bc \\ c & \longrightarrow & ac \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{B}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & cbca \\ b & \longrightarrow & bc \\ c & \longrightarrow & abc \end{array}$$

Ideal Whitehead Graph: Figure 6.27

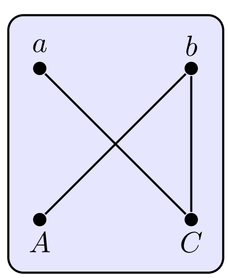

Figure 6.26. PNP: $n = 3,\ i = -1$

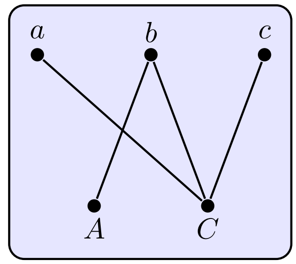

Figure 6.27. PNP: $n = 3$, $i = -1.5$

Example 6.25 $n = 2$, $i = -1$

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & baa \\ b & \longrightarrow & ba \end{array}$$

NP: $\dot{a}B$, $\dot{A}b\dot{a}$

PNP: Abb

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & ba \end{array}$$

NP: $abAB$

Example 6.26 $n = 4$, $i = -3$ *Perageometric*

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & adbadc \\ b & \longrightarrow & ababa \\ c & \longrightarrow & cab \\ d & \longrightarrow & abad \end{array}$$

NP: $\dot{A}BAdcadca\dot{b}$, $\dot{B}\dot{d}$ *PNP:* Ba

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & adbadc \\ b & \longrightarrow & ba \\ c & \longrightarrow & cadb \\ d & \longrightarrow & adbad \end{array}$$

NP: $DACDadba$

Example 6.27 $n = 4$, $i = -3$ *Geometric*

Representative with PNP:

$$\begin{array}{ccc} a & \longrightarrow & ba \\ b & \longrightarrow & baCb \\ c & \longrightarrow & ccdcABc \\ d & \longrightarrow & ccdcAB \end{array}$$

NP: $\dot{c}Aba$, $\dot{D}b$,$\dot{D}Cd\dot{c}$ *PNP:* $\dot{d}a\dot{C}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & baCab \\ c & \longrightarrow & cABcABdBAc \\ d & \longrightarrow & DbaCdBAc \end{array}$$

NP: $DbaCdBAc$

Example 6.28 *Geometric and Parageometric*

It is difficult to give a general form of geometric or parageometric examples of different ranks with or without PNP's, but these can be found experimentally for any particular rank. Thierry Coulbois's Train Track program can be used to generate a list of random examples for a particular rank, some of which will be geometric and parageometric.

6.2 NP but No PNP

Example 6.29 $n = 4$, $i = -2$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & aacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & a \\ d & \longrightarrow & bc \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bcd \\ c & \longrightarrow & ab \\ d & \longrightarrow & bc \end{array}$$

Ideal Whitehead Graph:Figure 6.28

Example 6.30 $n = 4$, $i = -.5$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & dacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & da \\ d & \longrightarrow & bd \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & da \\ b & \longrightarrow & bcd \\ c & \longrightarrow & dab \\ d & \longrightarrow & bd \end{array}$$

Ideal Whitehead Graph: Figure 6.29

Example 6.31 $n \geq 5$, $i = -2$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & aacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & a \\ d & \longrightarrow & ba_nc \\ e & \longrightarrow & d \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bcd \\ c & \longrightarrow & ab \\ d & \longrightarrow & ba_nc \\ e & \longrightarrow & d \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.30

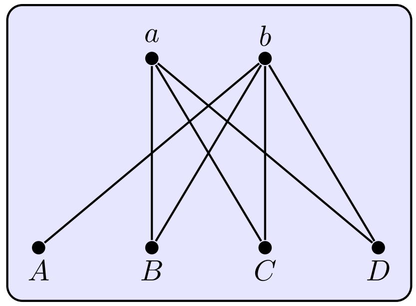

Figure 6.28. No PNP: $n = 4$, $i = -2$

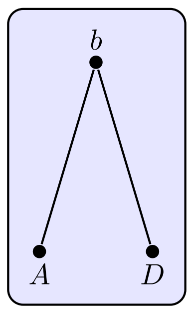

Figure 6.29. No PNP: $n = 4$, $i = -.5$

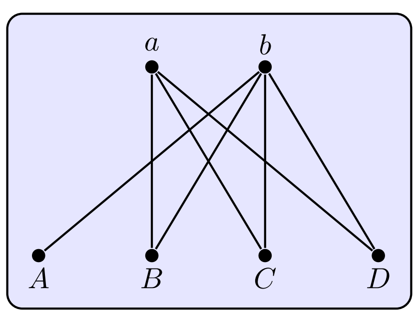

Figure 6.30. No PNP: $n \geq 5$, $i = -2$

Example 6.32 $n = 5,\ i = -3$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & cacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cac \\ d & \longrightarrow & dbe \\ e & \longrightarrow & b \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ca \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cabc \\ d & \longrightarrow & dbe \\ e & \longrightarrow & b \end{array}$$

Ideal Whitehead Graph: Figure 6.31

Example 6.33 $n \geq 6,\ i = -3$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & cacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cac \\ d & \longrightarrow & da_nbe \\ e & \longrightarrow & b \\ f & \longrightarrow & e \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ca \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cabc \\ d & \longrightarrow & da_nbe \\ e & \longrightarrow & b \\ f & \longrightarrow & e \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.32

Example 6.34 $n \geq 6,\ i = -2.5$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & cacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cad \\ d & \longrightarrow & ba_nbe \\ e & \longrightarrow & b \\ f & \longrightarrow & e \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ca \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cabd \\ d & \longrightarrow & ba_nbe \\ e & \longrightarrow & b \\ f & \longrightarrow & e \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.33

Example 6.35 $n \geq 5$, $i = -.5$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & dacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & da \\ d & \longrightarrow & ba_nd \\ e & \longrightarrow & d \\ f & \longrightarrow & de \\ & \vdots & \\ a_n & \longrightarrow & da_{n-1} \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & da \\ b & \longrightarrow & bcd \\ c & \longrightarrow & dab \\ d & \longrightarrow & ba_nd \\ e & \longrightarrow & d \end{array}$$

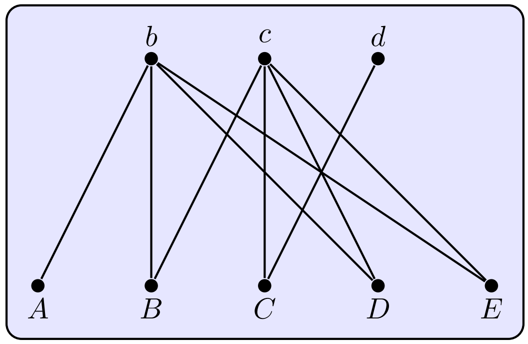

Figure 6.31. No PNP: $n = 5$, $i = -3$

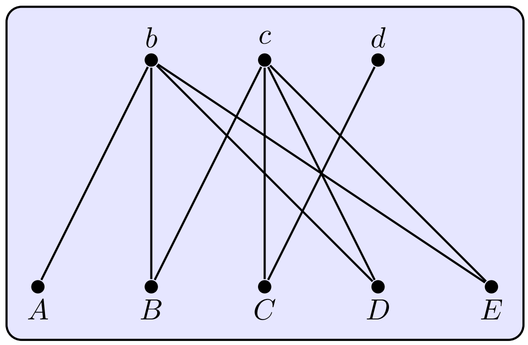

Figure 6.32. No PNP: $n \geq 6$, $i = -3$

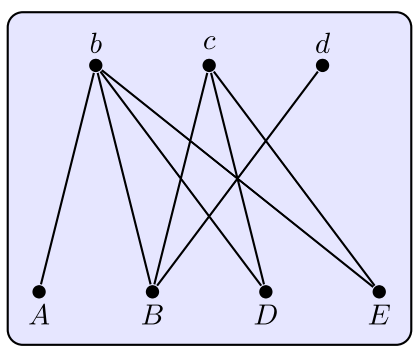

Figure 6.33. No PNP: $n \geq 6$, $i = -2.5$

$$\begin{array}{rcl} f & \longrightarrow & de \\ & \vdots & \\ a_n & \longrightarrow & da_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.34

Example 6.36 $n \geq 6$, $i = 2 - n$

Representative with NP but no PNP:

$$\begin{array}{rcl} a & \longrightarrow & aacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & a \\ d & \longrightarrow & bec \\ e & \longrightarrow & eca_n \\ f & \longrightarrow & fe \\ & \vdots & \\ a_n & \longrightarrow & a_n a_{n-1} \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{rcl} a & \longrightarrow & aba \\ b & \longrightarrow & bcd \\ c & \longrightarrow & ab \\ d & \longrightarrow & bec \\ e & \longrightarrow & eca_n \\ f & \longrightarrow & fe \\ & \vdots & \\ a_n & \longrightarrow & a_n a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.35

Example 6.37 $n \geq 7$, $6 \leq k < n$, $i = 2 - k$

Representative with NP but no PNP:

$$\begin{array}{rcl} a & \longrightarrow & aacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & a \\ d & \longrightarrow & bec \\ e & \longrightarrow & eca_k \\ f & \longrightarrow & fe \\ & \vdots & \\ a_k & \longrightarrow & a_k a_{k-1} \\ a_{k+1} & \longrightarrow & a_{k-1} \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bcd \\ c & \longrightarrow & ab \\ d & \longrightarrow & bec \\ e & \longrightarrow & eca_k \\ f & \longrightarrow & fe \\ & \vdots & \\ a_k & \longrightarrow & a_k a_{k-1} \\ a_{k+1} & \longrightarrow & a_{k-1} \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.36

Example 6.38 $n = 4$, $i = -1$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & aacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & a \\ d & \longrightarrow & bd \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bcd \\ c & \longrightarrow & ab \\ d & \longrightarrow & bd \end{array}$$

Ideal Whitehead Graph: Figure 6.37

Example 6.39 $n \geq 5$, $i = -1$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & aacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & a \\ d & \longrightarrow & ba_nd \\ e & \longrightarrow & d \\ f & \longrightarrow & e \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

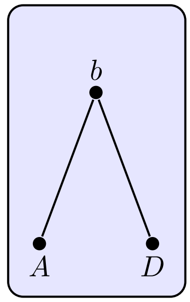

Figure 6.34. No PNP: $n \geq 5$, $i = -.5$

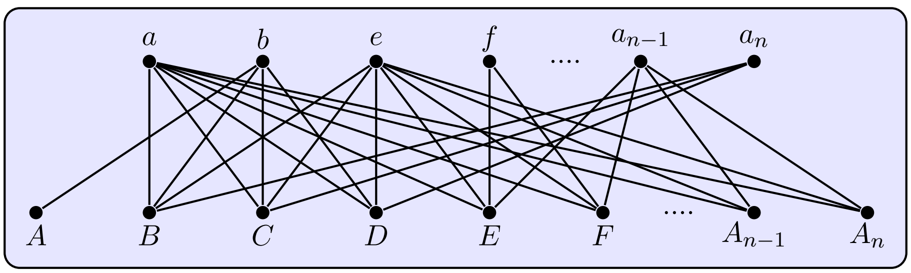

Figure 6.35. No PNP: $n \geq 6$, $i = 2 - n$

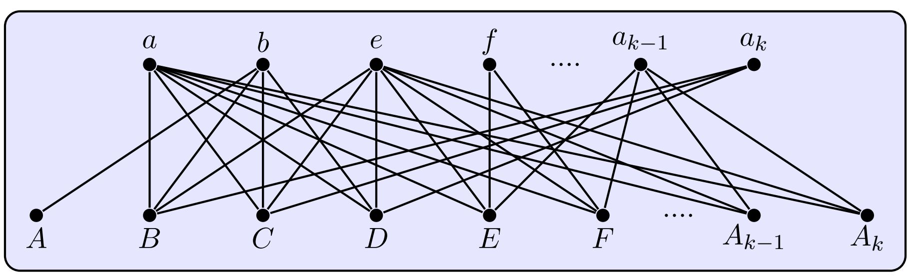

Figure 6.36. No PNP: $n \geq 7$, $6 \leq k < n$, $i = 2 - k$

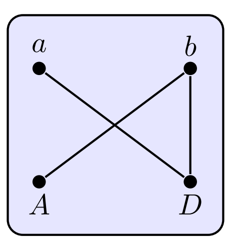

Figure 6.37. No PNP: $n = 4$, $i = -1$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bcd \\ c & \longrightarrow & ab \\ d & \longrightarrow & ba_n d \\ e & \longrightarrow & d \\ f & \longrightarrow & e \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.38

Example 6.40 $n = 4$, $i = -1.5$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & cacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cad \\ d & \longrightarrow & b \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ca \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cabd \\ d & \longrightarrow & b \end{array}$$

Ideal Whitehead Graph: Figure 6.39

Example 6.41 $n \geq 5$, $i = -1.5$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & cacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & ca_n ad \\ d & \longrightarrow & b \\ e & \longrightarrow & d \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ca \\ b & \longrightarrow & bcd \\ c & \longrightarrow & ca_n abd \\ d & \longrightarrow & b \\ e & \longrightarrow & d \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.40

Example 6.42 $n = 5$, $i = -2.5$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & cacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cad \\ d & \longrightarrow & dbe \\ e & \longrightarrow & b \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ca \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cabd \\ d & \longrightarrow & dbe \\ e & \longrightarrow & b \end{array}$$

Ideal Whitehead Graph: Figure 6.41

Example 6.43 $n \geq 6$, $i = 2.5 - n$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & cacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cad \\ d & \longrightarrow & dba_n \\ e & \longrightarrow & b \\ f & \longrightarrow & fe \\ & \vdots & \\ a_n & \longrightarrow & a_n a_{n-1} \end{array}$$

NP: $\dot{a}B$

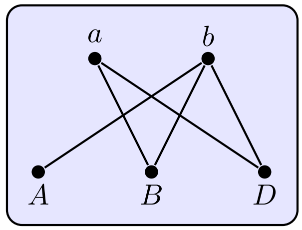

Figure 6.38. No PNP: $n \geq 5$, $i = -1$

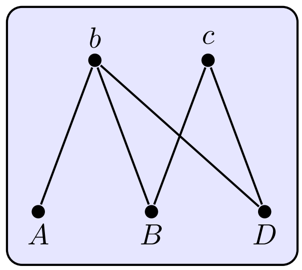

Figure 6.39. No PNP: $n = 4$, $i = -1.5$

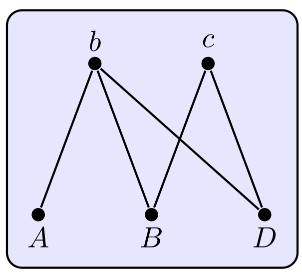

Figure 6.40. No PNP: $n \geq 5$, $i = -1.5$

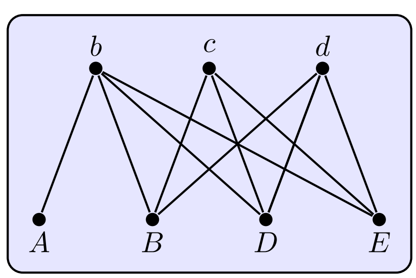

Figure 6.41. No PNP: $n = 5$, $i = -2.5$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ca \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cabd \\ d & \longrightarrow & dba_n \\ e & \longrightarrow & b \\ f & \longrightarrow & fe \\ & \vdots & \\ a_n & \longrightarrow & a_n a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.42

Example 6.44 $n \geq 7$, $6 \leq k < n$, $i = 2.5 - k$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & cacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cad \\ d & \longrightarrow & dba_k \\ e & \longrightarrow & b \\ f & \longrightarrow & fa_n e \\ g & \longrightarrow & gf \\ & \vdots & \\ a_k & \longrightarrow & a_k a_{k-1} \\ a_{k+1} & \longrightarrow & e \\ a_{k+2} & \longrightarrow & a_{k+1} \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ca \\ b & \longrightarrow & bcd \\ c & \longrightarrow & cabd \\ d & \longrightarrow & dba_k \\ e & \longrightarrow & b \\ f & \longrightarrow & fa_n e \\ g & \longrightarrow & gf \\ & \vdots & \\ a_k & \longrightarrow & a_k a_{k-1} \\ a_{k+1} & \longrightarrow & e \\ a_{k+2} & \longrightarrow & a_{k+1} \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.43

Example 6.45 $n = 4$, $i = -2.5$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & dacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & bc \\ d & \longrightarrow & a \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & daba \\ b & \longrightarrow & bcd \\ c & \longrightarrow & bc \\ d & \longrightarrow & ab \end{array}$$

Ideal Whitehead Graph: Figure 6.44

Example 6.46 $n \geq 5$, $i = 1.5 - n$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & aacd \\ b & \longrightarrow & bcd \\ c & \longrightarrow & bc \\ d & \longrightarrow & a_n da \\ e & \longrightarrow & de \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1}a_n \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bcd \\ c & \longrightarrow & bc \\ d & \longrightarrow & a_n dab \\ e & \longrightarrow & de \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1}a_n \end{array}$$

Ideal Whitehead Graph: Figure 6.45

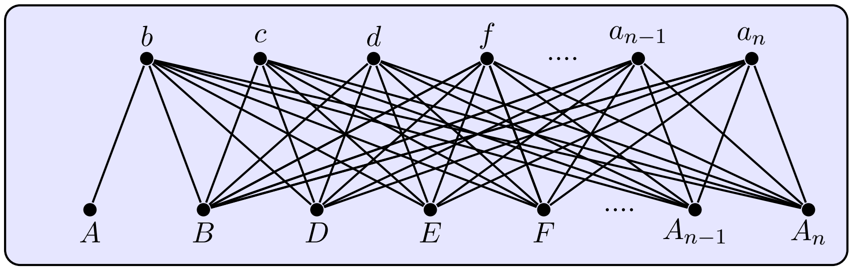

Figure 6.42. No PNP: $n \geq 6$, $i = 2.5 - n$

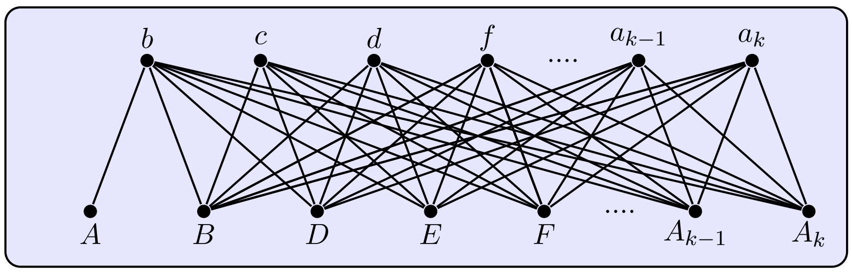

Figure 6.43. No PNP:$n \geq 7$, $6 \leq k < n$, $i = 2.5 - k$

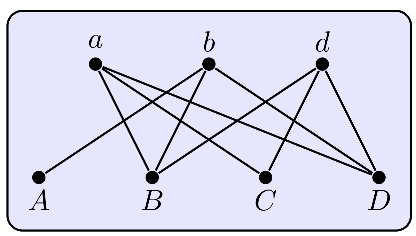

Figure 6.44. No PNP: $n = 4$, $i = -2.5$

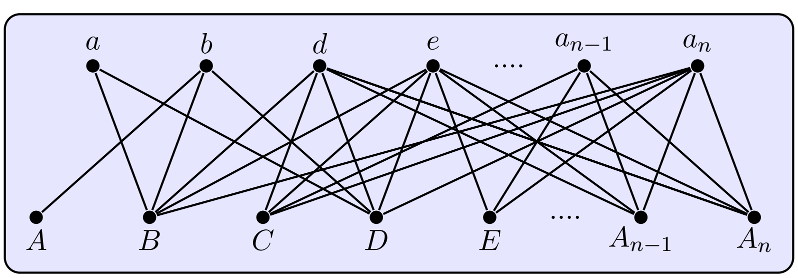

Figure 6.45. No PNP: $n \geq 5$, $i = 1.5 - n$

Example 6.47 $n = 3,\ i = -1.5$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & aabc \\ b & \longrightarrow & bbc \\ c & \longrightarrow & a \end{array}$$

NP: $\dot{a}B$

PNP: $\dot{c}\dot{B}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aba \\ b & \longrightarrow & bbc \\ c & \longrightarrow & ab \end{array}$$

Ideal Whitehead Graph: Figure 6.46

Example 6.48 $n = 3,\ i = -1$

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & cbaa \\ b & \longrightarrow & ba \\ c & \longrightarrow & bac \end{array}$$

NP: $\dot{a}B$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & cba \\ b & \longrightarrow & bab \\ c & \longrightarrow & babc \end{array}$$

Ideal Whitehead Graph: Figure 6.47

Example 6.49 $n = 2,\ i = -1,$ *Geometric*

Representative with NP but no PNP:

$$\begin{array}{ccc} a & \longrightarrow & ab \\ b & \longrightarrow & abb \end{array}$$

NP: $Ba\dot{b},\ a\dot{B}$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ab \\ b & \longrightarrow & bab \end{array}$$

NP: baBA

Example 6.50 $n = 4,\ i = -3,$ *Geometric*

Representative with NP but no PNP:

$$\begin{array}{rcc} a & \longrightarrow & abdacabda \\ b & \longrightarrow & dabdacadbacab \\ c & \longrightarrow & cab \\ d & \longrightarrow & dad \end{array}$$

NP: $Ba\dot{b}$, $cabCAD\dot{B}$

Stable Representative:

$$\begin{array}{rcc} a & \longrightarrow & adbdacadbda \\ b & \longrightarrow & bdacadbdacadb \\ c & \longrightarrow & cadb \\ d & \longrightarrow & dad \end{array}$$

NP: $cadbCADB$

Example 6.51 *$n3$, $i = -2$, Geometric*

Representative with NP but no PNP:

$$\begin{array}{rcc} a & \longrightarrow & ba \\ b & \longrightarrow & bbacb \\ c & \longrightarrow & cb \end{array}$$

NP: $A\dot{b}$, $cbCA\dot{B}$

Stable Representative:

$$\begin{array}{rcc} a & \longrightarrow & aba \\ b & \longrightarrow & bacab \\ c & \longrightarrow & cab \end{array}$$

NP: $cabCAB$

6.3 No NP

All the examples in this section have an Ideal Whitehead Graph with no cut vertices, so they can not have a representative with a NP.

Example 6.52 $n = 4$, $i = -1$

Stable Representative:

$$\begin{array}{rcc} a & \longrightarrow & ab \\ b & \longrightarrow & c \\ c & \longrightarrow & d \\ d & \longrightarrow & dac \end{array}$$

Ideal Whitehead Graph: Figure 6.48

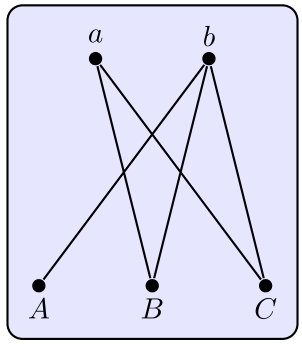

Figure 6.46. No PNP: $n = 3$, $i = -1.5$

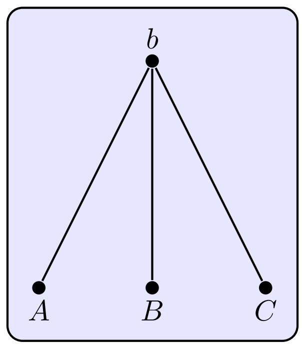

Figure 6.47. No PNP: $n = 3$, $i = -1$

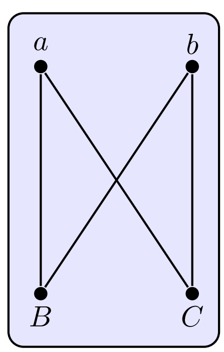

Figure 6.48. No NP: $n = 4$, $i = -1$

Example 6.53 $n = 5,\ i = -1$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ab \\ b & \longrightarrow & c \\ c & \longrightarrow & d \\ d & \longrightarrow & dec \\ e & \longrightarrow & ad \end{array}$$

Ideal Whitehead Graph: Figure 6.49

Example 6.54 $n \geq 6,\ i = -1$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ab \\ b & \longrightarrow & c \\ c & \longrightarrow & d \\ d & \longrightarrow & da_nc \\ e & \longrightarrow & ad \\ & \vdots & \\ a_n & \longrightarrow & aa_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.50

Example 6.55 $n = 4,\ i = -2$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ab \\ b & \longrightarrow & bd \\ c & \longrightarrow & b \\ d & \longrightarrow & dca \end{array}$$

Ideal Whitehead Graph: Figure 6.51

Example 6.56 $n = 5,\ i = -2$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ab \\ b & \longrightarrow & bd \\ c & \longrightarrow & b \\ d & \longrightarrow & ea \\ e & \longrightarrow & edca \end{array}$$

Ideal Whitehead Graph: Figure 6.52

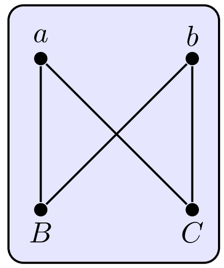

Figure 6.49. No NP: $n = 5$, $i = -1$

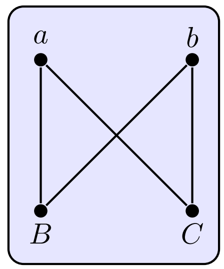

Figure 6.50. No NP: $n \geq 6$, $i = -1$

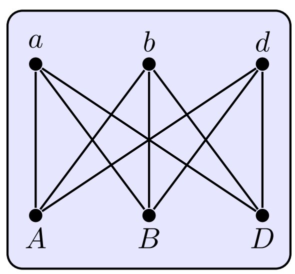

Figure 6.51. No NP: $n = 4$, $i = -2$

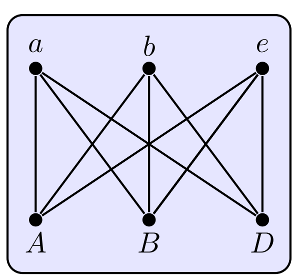

Figure 6.52. No NP: $n = 5$, $i = -2$

Example 6.57 $n \geq 6,\ i = -2$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ab \\ b & \longrightarrow & bd \\ c & \longrightarrow & b \\ d & \longrightarrow & da_n a \\ e & \longrightarrow & dca \\ f & \longrightarrow & e \\ & \vdots & \\ a_n & \longrightarrow & aa_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.53

Example 6.58 $n = 5,\ i = -3$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ab \\ b & \longrightarrow & bed \\ c & \longrightarrow & d \\ d & \longrightarrow & dca \\ e & \longrightarrow & ebe \end{array}$$

Ideal Whitehead Graph: Figure 6.54

Example 6.59 $n \geq 6,\ i = 2 - n$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aa_n \\ b & \longrightarrow & ba \\ c & \longrightarrow & d \\ d & \longrightarrow & dcb \\ e & \longrightarrow & ed \\ & \vdots & \\ a_n & \longrightarrow & a_n a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.55

Example 6.60 $n \geq 7,\ 6 \leq k < n,\ i = 2 - k$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aa_n \\ b & \longrightarrow & ba \\ c & \longrightarrow & d \\ d & \longrightarrow & dcb \end{array}$$

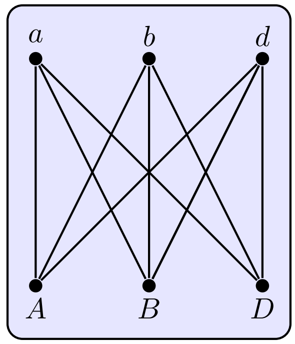

Figure 6.53. No NP: $n \geq 6$, $i = -2$

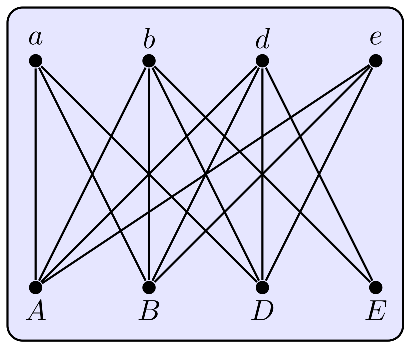

Figure 6.54. No NP: $n = 5$, $i = -3$

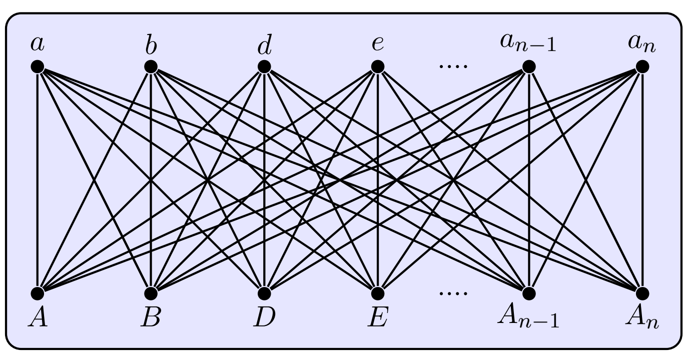

Figure 6.55. No NP: $n \geq 6$, $i = 2 - n$

$$\begin{array}{ccc} e & \longrightarrow & ed \\ & \vdots & \\ a_k & \longrightarrow & a_k a_n a_{k-1} \\ a_{k+1} & \longrightarrow & a_k \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.56

Example 6.61 $n = 4$, $i = -1.5$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ad \\ b & \longrightarrow & bc \\ c & \longrightarrow & a \\ d & \longrightarrow & ddba \end{array}$$

Ideal Whitehead Graph: Figure 6.57

Example 6.62 $n \geq 5$, $i = -1.5$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & aed \\ b & \longrightarrow & bc \\ c & \longrightarrow & a \\ d & \longrightarrow & ddba \\ e & \longrightarrow & d \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.58

Example 6.63 $n = 4$, $i = -2.5$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & bd \\ b & \longrightarrow & a \\ c & \longrightarrow & ca \\ d & \longrightarrow & dc \end{array}$$

Ideal Whitehead Graph: Figure 6.59

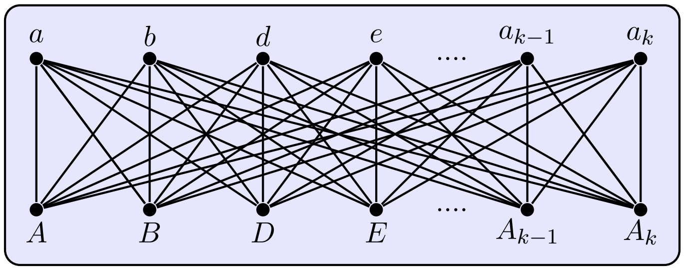

Figure 6.56. No NP:$n \geq 7$, $6 \leq k < n$, $i = 2 - k$

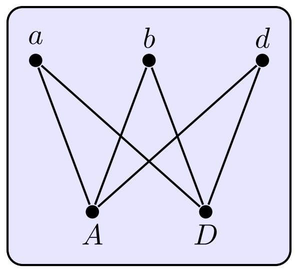

Figure 6.57. No PNP: $n = 4$, $i = -1.5$

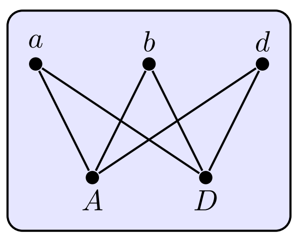

Figure 6.58. No NP: $n \geq 5$, $i = -1.5$

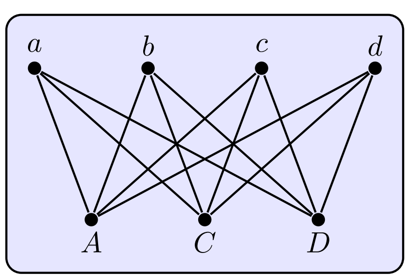

Figure 6.59. No NP: $n = 4$, $i = -2.5$

Example 6.64 $n \geq 5$, $i = -2.5$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & bd \\ b & \longrightarrow & a \\ c & \longrightarrow & ca \\ d & \longrightarrow & dec \\ e & \longrightarrow & d \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.60

Example 6.65 $n \geq 6$, $i = 1.5 - n$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & bd \\ b & \longrightarrow & e \\ c & \longrightarrow & ca \\ d & \longrightarrow & dca_nec \\ e & \longrightarrow & fe \\ & \vdots & \\ a_{n-1} & \longrightarrow & a_na_{n-1} \\ a_n & \longrightarrow & aa_n \end{array}$$

Ideal Whitehead Graph: Figure 6.61

Example 6.66 $n \geq 7$, $6 \leq k < n$, $i = 1.5 - k$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & bd \\ b & \longrightarrow & e \\ c & \longrightarrow & ca \\ d & \longrightarrow & dca_nec \\ e & \longrightarrow & fe \\ & \vdots & \\ a_{n-1} & \longrightarrow & a_na_{n-1} \\ a_n & \longrightarrow & aa_n \end{array}$$

Ideal Whitehead Graph: Figure 6.62

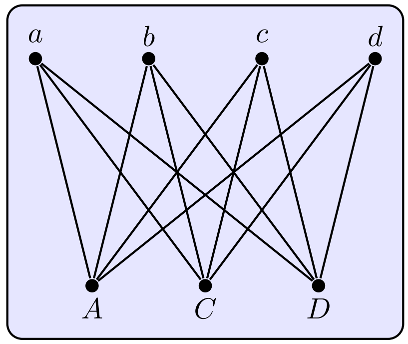

Figure 6.60. No NP: $n \geq 5$, $i = -2.5$

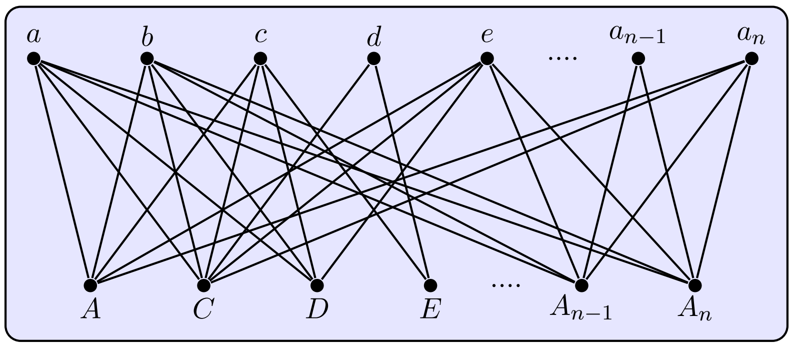

Figure 6.61. No NP: $n \geq 6$, $i = 1.5 - n$

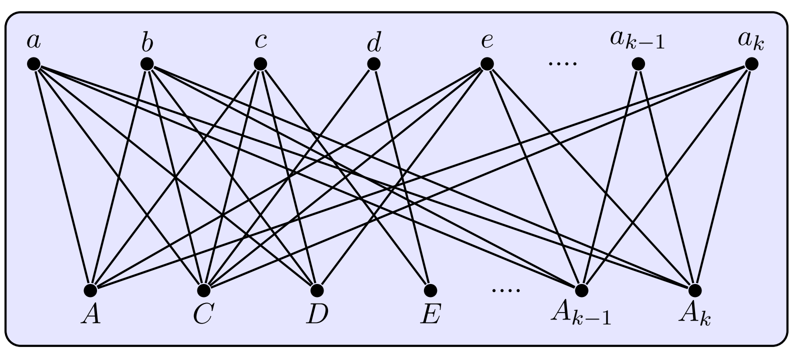

Figure 6.62. No NP: $n \geq 7$, $6 \leq k < n$, $i = 1.5 - k$

Example 6.67 $n = 5,\ i = -3.5$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ae \\ b & \longrightarrow & c \\ c & \longrightarrow & ba \\ d & \longrightarrow & dc \\ e & \longrightarrow & ed \end{array}$$

Ideal Whitehead Graph: Figure 6.63

Example 6.68 $n \geq 6,\ i = -3.5$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ae \\ b & \longrightarrow & c \\ c & \longrightarrow & ba \\ d & \longrightarrow & dc \\ e & \longrightarrow & eda_nd \\ f & \longrightarrow & e \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.64

Example 6.69 $n = 5,\ i = -3$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & ae \\ b & \longrightarrow & c \\ c & \longrightarrow & ca \\ d & \longrightarrow & dc \\ e & \longrightarrow & ebd \end{array}$$

Ideal Whitehead Graph: Figure 6.65

Example 6.70 $n \geq 6,\ i = -3$

Stable Representative:

$$\begin{array}{ccc}
a & \longrightarrow & aa_ne \\
b & \longrightarrow & c \\
c & \longrightarrow & ca \\
d & \longrightarrow & dc \\
e & \longrightarrow & ebd \\
f & \longrightarrow & e \\
 & \vdots & \\
a_n & \longrightarrow & a_{n-1}
\end{array}$$

Ideal Whitehead Graph: Figure 6.66

Example 6.71 $n = 4$, $i = -.5$

Stable Representative:

$$\begin{array}{ccc}
a & \longrightarrow & b \\
b & \longrightarrow & cA \\
c & \longrightarrow & dbc \\
d & \longrightarrow & c
\end{array}$$

Ideal Whitehead Graph: Figure 6.67

Example 6.72 $n = 4$, $i = 0$

Stable Representative:

$$\begin{array}{ccc}
a & \longrightarrow & Da \\
b & \longrightarrow & A \\
c & \longrightarrow & bc \\
d & \longrightarrow & c
\end{array}$$

Example 6.73 $n \geq 5$, $i = 0$

Stable Representative:

$$\begin{array}{ccc}
a & \longrightarrow & DA_na \\
b & \longrightarrow & A \\
c & \longrightarrow & bc \\
d & \longrightarrow & c \\
e & \longrightarrow & d \\
 & \vdots & \\
a_n & \longrightarrow & a_{n-1}
\end{array}$$

Example 6.74 $n = 3$, $i = 0$

Stable Representative:

$$\begin{array}{ccc}
a & \longrightarrow & ab \\
b & \longrightarrow & c \\
c & \longrightarrow & ac
\end{array}$$

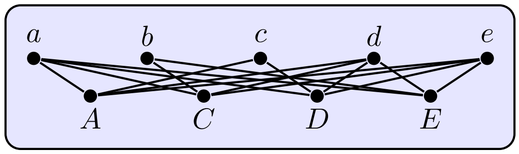

Figure 6.63. No NP: $n = 5$, $i = -3.5$

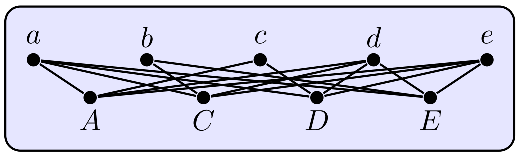

Figure 6.64. No NP: $n \geq 6$, $i = -3.5$

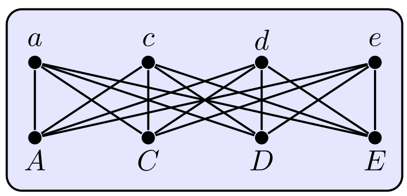

Figure 6.65. No NP: $n = 5$, $i = -3$

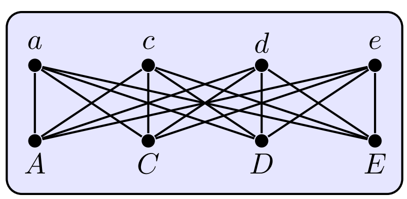

Figure 6.66. No NP: $n \geq 6$, $i = -3$

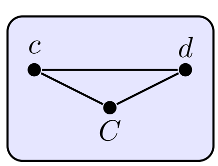

Figure 6.67. No NP: $n = 4$, $i = -.5$

Example 6.75 $n \geq 5$, $i = -.5$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & b \\ b & \longrightarrow & cA \\ c & \longrightarrow & da_nbc \\ d & \longrightarrow & c \\ e & \longrightarrow & d \\ f & \longrightarrow & e \\ & \vdots & \\ a_n & \longrightarrow & a_{n-1} \end{array}$$

Ideal Whitehead Graph: Figure 6.68

Example 6.76 $n = 3$, $i = -.5$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & AC \\ b & \longrightarrow & bcB \\ c & \longrightarrow & cabc \end{array}$$

Ideal Whitehead Graph: Figure 6.69

Example 6.77 $n = 3$, $i = -1.5$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & bac \\ b & \longrightarrow & a \\ c & \longrightarrow & b \end{array}$$

Ideal Whitehead Graph: Figure 6.70

Example 6.78 $n = 3$, $i = -1$

Stable Representative:

$$\begin{array}{ccc} a & \longrightarrow & bc \\ b & \longrightarrow & c \\ c & \longrightarrow & bca \end{array}$$

Ideal Whitehead Graph: Figure 6.71

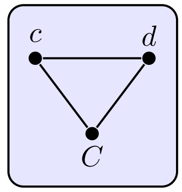

Figure 6.68. No NP: $n \geq 5$, $i = -.5$

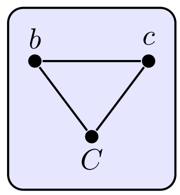

Figure 6.69. No NP: $n = 3$, $i = -.5$

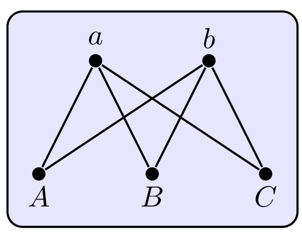

Figure 6.70. No NP: $n = 3$, $i = -1.5$

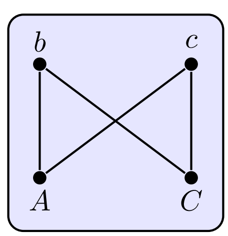

Figure 6.71. No NP: $n = 3$, $i = -1$

REFERENCES

[AK08] Y. Algor-Kfir, *Strongly contracting geodesics in Outer Space*, `arXiv:0812.1555v3`, 2008.

[Bes11] M. Bestvina, *A Bers-like proof of the existence of train tracks for free group automorphisms*, `arXiv:1001.0325v2`, 2011.

[Bes12] ———, *PCMI lectures on the geometry of Outer Space*, `http://www.math.utah.edu/pcmi12/lecture_notes/bestvina.pdf`, 2012.

[BF94] M. Bestvina and M. Feighn, *Outer limits*, `http://andromeda.rutgers.edu/~feighn/papers/outer.pdf`, 1994.

[BFH97] M. Bestvina, M. Feighn, and M. Handel, *Laminations, trees, and irreducible automorphisms of free groups*, GAFA **7** (1997), 215–244.

[BH92] M. Bestvina and M. Handel, *Train tracks and automorphisms of free groups*, Ann. Math. **135** (1992), 1–51.

[CV86] M. Culler and K. Vogtmann, *Moduli of graphs and automorphisms of free groups*, Invent. Math. **84** (1986), 91–119.

[FH09] M. Feign and M. Handel, *Abelian subgroups of* $Out(F_n)$, Geometry and Topology **13** (2009), 1657–1727.

[HM08] M. Handel and L. Mosher, *Axes in Outer Space*, `arXiv:1210.5762v1`, 2008.

[MH08] L. Mosher and M. Handel, *Parageometric outer automorphisms of free groups*, `arXiv:math/0410018v2`, 2008.

[Sta91] J. Stallings, *Folding of G-trees*, Arboreal group theory (R. Alperin, ed.), Math. Sci. Res. Insy. Publ., vol. 19, Springer, New York, 1991, ISBN 0-387-97518-7, pp. 355–368.

[Vog] K. Vogtmann, *Automorphisms of free groups and Outer Space*, `http://www.math.cornell.edu/~vogtmann/papers/Autosurvey/autosurvey.pdf`.